

BPS Explained I: Temperature Relaxation in a Plasma

or

How to Find the Coulomb Logarithm Exactly

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Abstract

This is the first of two lectures on the technique of dimensional continuation employed by Brown, Preston, and Singleton (BPS) to calculate such quantities as the charged particle stopping power and the temperature equilibration rate in a plasma. In this exposition we will examine some of the more basic points of dimensional continuation, with an emphasis on the Coulomb logarithm for electron-ion temperature equilibration. Dimensional continuation, or dimensional regularization as it is more properly known in quantum field theory, was originally developed as part of the renormalization procedure for the theories of the electroweak and other fundamental interactions in particle physics. Dimensional continuation is so general, in fact, that any *theory* can be unambiguously lifted to dimensions beyond three, and therefore the technique is powerful enough to apply in many other settings. The technique, however, is not well known outside the field theory and particle physics communities. This exposition will therefore be self-contained, intended for those who are not specialists in quantum field theory, and I will either derive or motivate any requisite field theory results or concepts. Of particular relevance is the analogy between the Coulomb logarithm as calculated by Lyman Spitzer on the one hand, and the Lamb shift as calculated by Hans Bethe on the other. While dimensional continuation is a well developed and a thoroughly tested method for regularizing any quantum field theory, BPS employs the method in a novel way that provides the leading and subleading behavior for processes that involve competing disparate energy or length scales. In particular, BPS calculated the temperature equilibration rate to leading and next-to-leading order in the plasma number density for any two species in a plasma that are in thermal equilibrium with themselves, but not necessarily with each other. No restriction is made on the charge, mass, or the temperature of the plasma species. It is, however, assumed that the plasma is not strongly coupled in the sense that the dimensionless plasma coupling parameter $g = e^2 \kappa_D / 4\pi T$ is small, where κ_D is the Debye wave number of the plasma. To leading and next-to-leading order in this coupling, the temperature equilibration rate is of the generic form $\Gamma = A g^2 \ln\{Cg\}$. The precise numerical coefficient A in front of the logarithm has been known for some time, while BPS have recently computed the constant C under the logarithm. It should be emphasized that the BPS result is not a model, but rather it is an exact calculation of the leading terms in a well-defined perturbation theory. This exact result differs from approximations and models given in the literature.

Contents

I. Introduction and Context	1
A. The Problem	2
B. The Problem with the Problem	5
II. Traditional Methods	6
A. Heuristic Models	6
B. Convergent Kinetic Equations: Traditional Approach	8
III. Bessel Function Example	10
A. Analogy with Dimensional Continuation	10
B. Leading Order Terms	11
C. Some Comments on Analytic Continuation	13
D. Next-to-Leading Order Term	15
E. Assembling the Pieces	16
IV. Dimensional Continuation	18
A. Rate of Energy Exchange as a Perturbative Expansion	18
1. The g -Expansion	18
2. Next-to-Leading Order and the Coulomb Logarithm	19
3. Factors of g Inside the Coulomb Logarithm	20
B. Mathematics of Dimensional Continuation	21
C. Physics of Dimensional Continuation	25
V. The Lamb Shift and the Coulomb Logarithm	30
VI. Calculating the Rate Systematically with Dimensional Continuation	35
A. Dimensional Reduction of BBGKY	35
B. Calculating the Leading Order Term	38
C. Next-to-Leading Order from Leading Order via Analytic Continuation	39
D. Returning to Three Dimensions	40
VII. Some Closing Remarks	41
A. Summary	41
B. Further Context	42
Acknowledgments	43
References	43

I. INTRODUCTION AND CONTEXT

This is the first of two lectures on a new technique for calculating the temperature equilibration rate between electrons and ions in a weakly to moderately coupled fully-ionized plasma, exact to leading and next-to-leading order in the plasma number density. This calculation was first performed in Section 12 of Ref. [1], a work whose primary focus was the charged particle stopping power in a plasma.¹ This paper assumed familiarity with a number of field theory concepts, and Section 12 relied heavily on the charged particle stopping power results derived in previous sections of that work. In contrast, these lectures will be self-contained. I will either derive or motivate the requisite field theory background for a complete reading of Ref. [1], with an emphasis on some of the subtleties of the calculational techniques and the concepts behind them (this lecture). More to the point, since the rate calculation stands on its own, it should be presented on its own (the following lecture). Indeed, since the calculation of the rate is somewhat less involved than that of the stopping power, it more clearly illustrates the tools and concepts imported from field theory.

In addition to clarifying the method of dimensional continuation, this lecture will place Ref. [1] in the context of more familiar and traditional approaches to the rate problem. In particular, I will show that dimensional continuation can be viewed as a systematic implementation of the approach based on convergent kinetic equations. Finally, in the next lecture, I will go on to derive the main result from Section 12 of Ref. [1], the rate coefficient (1.3) of this lecture. By working in the Born approximation, and adopting the methods of Ref. [3], I will derive this result in a much simpler manner than originally presented in Ref. [1]. While Lectures I and II are self-contained, they are complementary and should be read as a unit.²

The strategy employed by Ref. [1], hereafter referred to as BPS, consists of two steps. First, we will find a dimensionless parameter in which to perform a *controlled* perturbative expansion of the rate, expanding to leading and next-to-leading order in this parameter.

¹ For a short cursory version of this work, see Ref. [2].

² In other words, Lecture I (this lecture) consists of the basic theory and techniques behind dimensional continuation, while Lecture II (next lecture) will be a specific calculation in the extreme quantum limit: only by performing a calculation can one understand the underlying ideas of the calculation. Ideally, I would then like to continue these first two lectures with three additional ones. In Lecture III, I would present the full calculation of the temperature relaxation rate performed in Ref. [1], valid beyond the Born approximation. The full calculation is accurate for a weakly to moderately coupled plasma in both the classical and quantum limits, and any regime in-between, regardless of the mass and temperature difference of the plasma species. I would then present the details of the full quantum corrections in Lecture IV. Finally, in Lecture V, I would come full circle, simplifying the general equilibration rate calculated in Lectures III and IV to obtain the extreme quantum limit of Lecture II, equation (1.3) of this lecture. This would provide two independent calculations of the rate (1.3), but alas, time does not permit these last three supplements.

Second, we will deploy a technique from quantum field theory that will allow us to calculate the coefficients of these leading and subleading terms *exactly*. The exact leading order term is not very difficult to find, and has been known since the classic work of Spitzer. The next-to-leading order term, on the other hand, was not known exactly until the recent BPS calculation. The third-order term provides an estimate of the error of the calculation. When the plasma is weakly to moderately coupled, the error will be small and the rate will be approximated quite accurately by the first two terms of this expansion.

To calculate the expansion coefficients, we will exploit a field theory technique known as dimensional regularization (or dimensional continuation, as I will call it here). This application of dimensional continuation is quite different from its intended purpose in the renormalization procedure of quantum field theory. Dimensional continuation was originally developed as an elegant regularization scheme in which the fundamental symmetries of a field theory could be maintained while still rendering finite the otherwise infinite integrals that arise when calculating Feynman diagrams. I will show how this technique can be used in a novel fashion to extract next-to-leading order physics that has, until now, remained inaccessible. In other words, I will show how dimensional continuation provides an exact result for the corresponding Coulomb logarithm of the process in question. I will also take the opportunity to correct a small algebra mistake for the electron-ion equilibration rate presented in Section 12 of Ref. [1].

A. The Problem

The general formalism starts with a plasma composed of multiple species labeled by an index b , the various species being delineated by of a common electric charge e_b and a common mass m_b .³ Each species is assumed to be in thermal equilibrium with itself at temperature T_b with a spatially uniform number density n_b . I will drop the subscript on the charge of the electron and write $e_{\text{electron}} = -e$ (with $e > 0$), although the electron mass will be denoted by m_e , and the number density and temperature of the electron plasma component by n_e and T_e respectively. I will use a lower case subscript i to denote a single ion species of charge $e_i = Z_i e$, mass m_i , number density n_i , and temperature T_i . I will

³ The final constraint of BPS is that the ionization state of each component species does not change. In other words, the charges can be expressed as $e_b = Z_b e$ with Z_b fixed. While this simplifying assumption has its limitations, it facilitates the analytic calculation of the stopping power and temperature equilibration rate. However, one can take the Z_b to be fractional to mock-up ionization in a simple manner. Or better yet, it should be possible to combine the BPS results with models of the ionization effects. For a hot low- Z plasmas, such as a deuterium-tritium plasma during ICF ignition, the ions are likely to be fully ionized in any event, so this is not a serious restriction for clean thermonuclear burn with low- Z impurities.

employ a capital-I subscript to denote properties that correspond to the collective set of ions, such as the total ion number density $n_I = \sum_i n_i$ or a common ion temperature T_I . When equilibrium distributions are required in calculations, they are assumed to be Maxwell-Boltzmann, although a generalization to Fermi-Dirac statistics can be accomplished with more effort [3]. For problems involving hot thermonuclear burn, however, the fugacity is small and Maxwell-Boltzmann statistics is an accurate approximation.

Let $d\mathcal{E}_{ab}/dt$ denote rate at which the energy density of plasma species a changes because of its Coulomb interactions with another species b (the rate *from* the a -species *to* the b -species). This rate is proportional to the temperature difference, and can be expressed by

$$\frac{d\mathcal{E}_{ab}}{dt} = -\mathcal{C}_{ab} (T_a - T_b) . \quad (1.1)$$

The sign convention in (1.1) implies that when the rate coefficients \mathcal{C}_{ab} are positive, then energy will flow from the hotter species to the cooler species, as it must. Section 12 of BPS used dimensional continuation to calculate the general rate coefficients \mathcal{C}_{ab} in a weakly coupled, but otherwise arbitrary, plasma. For simplicity, we will not perform the general calculation until Lecture III. In this and the following lecture, we will concentrate on the energy exchange between electrons and ions only. Since the electron mass m_e is so much smaller than a typical ion mass m_I , the electrons will come into equilibrium first with temperature T_e on some time scale τ_e . The energy transfer rate among ions is a factor $\sqrt{m_e/m_I}$ slower than the corresponding rate for electrons, and therefore the ions will equilibrate to a common temperature T_I in a time $\tau_I \sim \tau_e \sqrt{m_I/m_e}$. Finally, as the electrons and ions exchange energy through Coulomb interactions, these systems too will equilibrate on a time scale $\tau_{eI} \sim \tau_I \sqrt{m_I/m_e} \sim \tau_e (m_I/m_e)$. Consequently, one finds a hierarchy of time scales $\tau_e \ll \tau_I \ll \tau_{eI}$, and it indeed makes sense to consider the electron and ion systems as having distinct temperatures T_e and T_I , with subsequent equilibration between them. Taking $a = e$ and $b = i$ in (1.1), and since the ions have a common temperature $T_i = T_I$, the rate equation of interest is obtained by summing over the ion components of (1.1) to give

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI} (T_e - T_I) , \quad (1.2)$$

where $\mathcal{C}_{eI} = \sum_i \mathcal{C}_{ei}$ and $d\mathcal{E}_{eI}/dt = \sum_i d\mathcal{E}_{ei}/dt$.

The coefficient \mathcal{C}_{eI} is the quantity we wish to calculate in this and the next lecture. This coefficient contains the energy-exchange physics between electrons and ions resulting from mutual Coulomb interactions, including possible collective effects and large-angle collisions. General expressions for the individual \mathcal{C}_{ei} were calculated in Section 12 of BPS. They are somewhat complicated and involve various one-dimensional integrals that can only be performed numerically. However, the collective rate coefficient \mathcal{C}_{eI} simplifies considerably when

the mild restriction $m_e/T_e \ll m_I/T_I$ is imposed (a sum-rule is employed in the approximation, and the simplification occurs only for $\mathcal{C}_{eI} = \sum_i \mathcal{C}_{ei}$ and not for the individual \mathcal{C}_{ei}). If the high temperature limit is further imposed, then the rate can be written in a quite simple analytic form.

For $m_e/T_e \ll m_I/T_I$ and $T_{e,I} \gg \epsilon_H$:

$$\mathcal{C}_{eI} = \frac{\kappa_e^2}{2\pi} \omega_I^2 \sqrt{\frac{m_e}{2\pi T_e}} \ln \Lambda_{\text{BPS}} , \quad \text{with} \quad \ln \Lambda_{\text{BPS}} = \frac{1}{2} \left[\ln \left\{ \frac{8T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right] , \quad (1.3)$$

where $\gamma = 0.57721 \dots$ is the Euler constant, κ_e and ω_e are the electron Debye wave number and plasma frequency, and $\omega_I^2 = \sum_i \omega_i^2$ is sum of the squares of the ion plasma frequencies. Since the small binding energy $\epsilon_H = 13.6 \text{ eV}$ of the hydrogen atom sets the temperature scale, and since the condition $m_e/T_e \ll m_I/T_I$ is not very restrictive, the rate coefficient (1.3) is applicable in almost all circumstances of interest. We shall devote the next lecture to deriving this expression. For now, note that equation (1.3) corresponds to Eqs. (3.61) and (12.12) of Ref. [1], where I have taken this opportunity to correct a small transcription error: when passing from Eq. (12.43) to Eq. (12.44) in Ref. [1], a factor of 1/2 was dropped. Restoring this factor of 1/2 changes the additive constant outside the logarithm from the $-\gamma - 2$ that appears in Eq. (12.12) of Ref. [1] to the constant $-\gamma - 1$ in (1.3) above.

For reasons to be discussed shortly, rationalized units are preferred for dimensional continuation, and I will employ this choice in all that follows. Nonetheless, expression (1.3) is written in a manner that does not depend upon this choice: the Debye wave number κ_e , and the plasma frequencies ω_e and ω_i can be calculated in your favorite units. For example, in Gaussian units where the electric potential takes the form $V = e^2/r$, the Debye wave number and the plasma frequency of species b are given by $\kappa_b^2 = 4\pi e_b^2 n_b/T_b$ and $\omega_b^2 = 4\pi e_b^2 n_b/m_b$. In the rationalized units employed here, the electric potential is given by $V = e^2/4\pi r$, and we have

$$\kappa_b^2 = \frac{e_b^2 n_b}{T_b} \quad (1.4)$$

$$\omega_b^2 = \frac{e_b^2 n_b}{m_b} . \quad (1.5)$$

The square of the total Debye wave number is $\kappa_D^2 = \sum_b \kappa_b^2$, and the total Debye wave length is $\lambda_D = \kappa_D^{-1}$.

B. The Problem with the Problem

Let us now consider an arbitrary plasma component of mass m , which I will otherwise leave unspecified, and let $f(\mathbf{v}, t)$ denote the Boltzmann distribution for this species. Then the average (kinetic) energy density of this component is

$$\mathcal{E} = \int d^3v \frac{1}{2} m v^2 f(\mathbf{v}, t) . \quad (1.6)$$

If we work to leading and next-to-leading order in the number density, calculating the energy exchange between plasma components will then involve keeping a tally only of the kinetic energy, as in (1.6). This is because the potential energy is higher order in the number density [or more precisely, the potential energy is higher order in the plasma coupling g , to be defined later in (4.2)]. As the system interacts with other plasma components through mutual Coulomb interactions, it will lose or gain energy depending on the temperature gradients with other species, and the energy exchange rate is given by

$$\frac{d\mathcal{E}}{dt} = \int d^3v \frac{1}{2} m v^2 \frac{\partial f}{\partial t}(\mathbf{v}, t) . \quad (1.7)$$

In contrast to (1.1) and (1.2), for ease of notation I have temporarily dropped the plasma component subscripts on the rate, and I will keep with this convention until the final calculation presented in Section VIB. We see that the entire problem is bound up in calculating the rate of change $\partial f / \partial t$ from an appropriate kinetic equation that captures the relevant physics. As it turns out, however, there is a serious problem in performing all such calculation with the Coulomb potential in three dimensions: the integrals in the kinetic equations diverge logarithmically, and they do so at both large and small distances.⁴

For processes in which large-angle scattering is important, such as the charged particle stopping power, it is natural to use the Boltzmann equation, which I will write in the abbreviated form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = B[f] , \quad (1.8)$$

where ∇ is the gradient in position space, and $B[f]$ is the scattering kernel, whose precise form will not concern us until the next lecture. The gradient vanishes because of spatial uniformity, so we will set $\mathbf{v} \cdot \nabla f = 0$. The Boltzmann equation was designed to account for the statistical effects of short-distance collisions, and although its original context was classical, quantum two-body scattering effects can easily be incorporated. In fact, since the

⁴ It is curious that this problem occurs for the Coulomb potential only in three dimensions, the case of most physical relevance in plasma physics.

scattering phase shifts δ_ℓ are known analytically for the Coulomb potential, Ref. [1] used this to calculate the two-body quantum corrections to all orders [in the quantum parameter η to be defined in (4.14)]. The kernel $B[f]$ therefore contains all short-distance or ultraviolet physics, for both classical and quantum scattering. However, (in three spatial dimensions) the Coulomb potential is long-range, and the integrals in $B[f]$ diverge in the infrared; or equivalently, if we write the scattering kernel $B[f]$ in terms of momentum integrals, the divergence appears at small values of momentum. This was not a problem in Boltzmann's original formulation of (1.8), since the Coulomb potential was unknown at that time, and he modeled particle collisions in terms of hard-sphere scattering. In summary, the Boltzmann equation gets the short-distance physics correct, including quantum two-body scattering, but it misses the infrared physics. The fact that the Boltzmann equation misses the long-distance physics manifests itself as an infrared divergence in the scattering kernel $B[f]$, thereby rendering calculations meaningless (unless we tame, or regularize, this divergence).

Given that the Boltzmann equation misses the long-distance or infrared (IR) physics, we might be tempted to try the Lenard-Balescu equation, which I will write in the abbreviated form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = L[f] , \quad (1.9)$$

where $L[f]$ is a scattering kernel whose exact form will be needed only in the next lecture. Again, the gradient term will be set to zero because of spatial uniformity. The Lenard-Balescu equation takes the form of a Fokker-Plank equation, with the kernel $L[f]$ chosen to capture the correct IR physics. However, for the Coulomb potential (in three spatial dimensions), the Lenard-Balescu equation misses the short-distance or ultraviolet (UV) physics, and this is manifested by a UV divergence in $L[f]$. The situation for the Lenard-Balescu equation is exactly reversed compared to that of the Boltzmann equation. This is what Ref. [4] calls the “complementarity” of these two kinetic equations, and in the dimensional continuation procedure we will use this fact to our advantage.

II. TRADITIONAL METHODS

A. Heuristic Models

The rate equation (1.7) reduces to a one-dimensional integral over the entire range of physical length scales (or momentum scales, if one so chooses), from zero all the way to infinity. Trouble arises for the Coulomb potential in three dimensions since the integral in question is logarithmically divergent at both integration limits. We must therefore regulate

the integral in some manner. Dimensional continuation is one such procedure, but there are others. This divergence problem was first worked around by simply cutting off the divergent integrals by hand, with the cutoffs themselves being chosen by physical arguments (rather than a calculation).

The energy exchange rate we are considering is but an example of a larger class of problems involving characteristic, but disparate, length or energy scales in which the measured quantity of interest is (logarithmically) insensitive to the physics above and below these scales. For these problems, the simplest and most intuitive regularization scheme is to replace the offending integration limits, *i.e.* infinity and zero, by the finite and non-zero physical scales of the problem. These two scales then act as formal integration cutoffs, giving a finite logarithm of the ratio of the scales. Furthermore, because the system is insensitive to the physics above and below the respective cutoffs, this procedure provides a physically meaningful result. Expressed in terms of length, we will denote the long- and short-distance scales by b_{\max} and b_{\min} , and the integral over scales leads to a finite logarithm involving the ratio of the physical length scales, so that

$$\frac{d\mathcal{E}}{dt} = K \ln \left\{ \frac{b_{\max}}{b_{\min}} \right\} , \quad (2.1)$$

where K is an easily determined prefactor with dimensions of energy density per unit time. As we shall see, a calculation to leading order in the number density is sufficient to provide the coefficient K , while a next-to-leading order calculation is required to find the exact terms under the logarithm.

The problem with this regularization prescription, which I will call the *heuristic scheme*, is that we can only estimate the values of the physical scales b_{\max} and b_{\min} to within factors of order one or so. For example, it is physically reasonable that the scale of the long distance cutoff in a plasma is set by a Debye length, so that $b_{\max} = c \kappa_D^{-1}$ with c being a dimensionless constant of order unity; but what determines the exact value of c ? In fact, how does $b_{\max} \sim \kappa_D^{-1}$ arise naturally from the kinetic equations themselves, rather than simply being chosen by hand? And should one use the total Debye wave number κ_D , or just the contribution from the electrons κ_e ? The origin of the short distance cutoff b_{\min} is even less clear. In the extreme classical limit, we expect this scale to be set by the classical distance of closest approach r_{cl} between two colliding particles, so that $b_{\min} \sim r_{\text{cl}}$.⁵ On the other hand, we expect quantum effects to dominate when r_{cl} becomes smaller than some thermal Compton wavelength $r_{\text{qm}} \sim \hbar/q$, with q being a typical thermally averaged momentum transfer. In

⁵ A further ambiguity arises in precisely defining r_{cl} , as this involves a somewhat arbitrary choice in the thermal averaging procedure.

this case we expect $b_{\min} \sim r_{\text{qm}}$. Worse yet, the intermediate region in which neither classical nor quantum processes dominate is often realized in a weakly coupled plasma, and in this case it is even more obscure how one should choose b_{\min} . We must interpolate between the extreme classical and the extreme quantum scales, but in the literature the exact procedure is always somewhat *ad hoc*. For example, a common choice is to define b_{\min}^2 to be the sum of the squares of the classical and quantum scales, so that

$$b_{\min} = \left(r_{\text{cl}}^2 + r_{\text{qm}}^2 \right)^{1/2} . \quad (2.2)$$

I will have more to say about this in Section IV A 3.

The heuristic scheme forces us to choose the specific forms of b_{\max} and b_{\min} motivated by imprecise physical arguments or heuristic exercises, which leads us into the art of model building rather than systematic calculation. Indeed, the very notation that we must *choose* a cutoff is misleading, since the physics itself must conspire to render all integrals finite. Consequently, the heuristic method suffers from an unknown coefficient under the logarithm, and only the approximate value of the ratio b_{\max}/b_{\min} can be determined with this method (in fact, this ratio varies across an order of magnitude over models in the literature, rather than factors of two or three). As we shall see, determining the constant under the logarithm exactly is equivalent to determining the next-to-leading order term exactly; therefore, models of the form (2.1) are accurate only to leading order, and no better.

B. Convergent Kinetic Equations: Traditional Approach

Rather than merely regulating the *integrals* in a rate derived from the kinetic equations, as with (2.1), a more sophisticated approach involves regularizing the divergences in the *kinetic equations* themselves. In other words, the *theory* itself is regularized, rather than a particular quantity being calculated within the theory. The method of dimensional continuation falls into this category, albeit with somewhat more subtle mathematical machinery than traditional approaches. These approaches, of which Refs. [5, 6, 7] are good examples, are summarized and placed into a common framework by Aono in Ref. [4]. As discussed in Section IB of this lecture, one can view the Boltzmann and Leonard-Balescu equations as providing complementary physics since they both succeed and fail in complementary regimes. The Boltzmann equation gets the short-distance physics correct, while the Leonard-Balescu equation captures the long-distance physics; conversely, Boltzmann and Leonard-Balescu miss the long- and short-distance physics, respectively. This complementarity motivates a class of kinetic equations of the form [4]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = B[f] + L[f] - R[f] , \quad (2.3)$$

where $R[f]$ is a carefully chosen “regulating kernel” designed to subtract the long-distance divergence of the Boltzmann equation and the short-distance divergence of the Lenard-Balescu equation. At the same time, the kernel $R[f]$ must preserve the correct short-distance physics of the Boltzmann equation and the correct long-distance physics of the Lenard-Balescu equation (a minimal requirement of the regulating kernel $R[f]$ is that it take a Hippocratic Oath to *do no harm*, at least to subleading order in the plasma coupling). Each term on the right-hand-side of (2.3) is separately divergent, but collectively they lead to a finite collision kernel if properly interpreted.⁶ As an example, Gould and DeWitt [7] regulated the long-distance divergence of the Boltzmann equation by simply replacing the Coulomb potential with a Debye screened Yukawa-like potential,

$$V_{\text{debye}}(r) = \frac{e^2}{4\pi r} \mathfrak{e}^{-\kappa_D r} , \quad (2.4)$$

where, as not to confuse symbols, I write \mathfrak{e} as the base of the natural logarithm and e as the electric charge. While Ref. [7] performed this operation by hand using physical arguments, one could easily introduce a kernel $R[f]$ to do the same.

While the approach to convergent kinetic equations described by Aono might appear to be more rigorous than the aforementioned model building approach of (2.1), it is no more systematic: methods based on (2.3) or its equivalent do not contain the ability to estimate their own error, *i.e.* they cannot determine their domain of applicability. There is nothing in the formalism of (2.3) that keeps track of the plasma coupling constant, or the order to which we are working in this constant. Indeed, one does not generally think of (2.3) in terms of a perturbation theory. In contrast, Ref. [1] calculates the rate using a systematic expansion in the plasma number density, or more precisely, in a dimensionless plasma coupling parameter g [to be defined by (4.2) and discussed at length in Section IV A]. Although written in a disguised form, the BPS rate coefficient (1.3) is an expansion to leading and next-to-leading order in the coupling parameter: the leading order term goes like $g^2 \ln g$, the next-to-leading order is proportional to g^2 , and the $\mathcal{O}(g^3)$ term provides an estimate of the error. Translating the work of Gould and DeWitt [7] into the language of a perturbative expansion in a plasma coupling constant, it turns out that their result is valid to order g^2 and is in agreement with

⁶ By “properly interpreted” I really mean that each term on the right-hand-side of (2.3) should be separately regularized in some manner, rendering the individual kernels $B[f]$, $L[f]$, and $R[f]$ separately finite: $B[f]$ must be regulated at long-distances, $L[f]$ at short-distances, and $R[f]$ at both long- and short-distances. If this procedure is executed correctly, so that the long-distance divergences of $R[f]$ and $B[f]$ cancel, as do the short-distance divergences of $R[f]$ and $L[f]$, both in a consistent fashion, then the entire right-hand-side of (2.3) remains finite as the cutoffs are removed. If $R[f]$ does not disturb the $\mathcal{O}(g^2)$ physics, then the convergent kinetic equation will be accurate to $\mathcal{O}(g^2)$. I will have more to say about how one regulates long- and short-distances *consistently* in Section V, and how this can be a quite non-trivial process.

BPS to this order.⁷ However, in their final result, Gould and DeWitt retain spurious higher order terms in g^3 . I call these terms “spurious” because Ref. [7] did not calculate the full set of order g^3 terms, but only some of these terms. Indeed, the notion of a systematic expansion in a small dimensionless parameter does not enter their calculational framework. As this example shows, convergent kinetic equations can be more accurate than the heuristic model building technique of the previous section, but one cannot be sure of their accuracy until a comparison with a systematic calculation has been made, as with BPS and Ref. [7].

III. BESSEL FUNCTION EXAMPLE

A. Analogy with Dimensional Continuation

I will illustrate the main points of dimensional continuation with an example involving the modified Bessel function $K_0(x)$, with an emphasis on analytic continuation and how this can be used to extract leading and next-to-leading order behavior. This example was first presented in Ref. [8], and for pedagogical purposes it was also included in Appendix A of Ref. [1]. This example contains all the essential features of dimensional continuation, but in a mathematically simple form, and while it is an imperfect analogy, as all analogies are, it is explicit in all its details. We will show that the modified Bessel function $K_0(x)$ has the expansion

$$K_0(x) = \underbrace{-\ln x}_{\text{LO}} + \underbrace{\ln 2 - \gamma}_{\text{NLO}} + \mathcal{O}(x^2) = -\ln\left(\frac{e^\gamma}{2} x\right) + \mathcal{O}(x^2), \quad (3.1)$$

to leading order (LO) and next-to-leading order (NLO) in x , with $\gamma = 0.577216 \dots$ denoting Euler’s constant. This expansion is quite accurate for small values x , with an error of order x^2 rather than x for symmetry reasons. The asymptotic expansion (3.1) is a well known result [9], but it is rather difficult to prove by conventional methods because of the non-analytic leading-log behavior. However, the method of dimensional continuation allows us to derive this result rather easily. The price one pays for this ease of derivation is that one must learn (or recall) a bit of mathematical machinery which, at first sight, seems unrelated to the problem at hand.

We start with the general integral representation of the modified Bessel functions [10],

$$K_\nu(x) = \frac{1}{2} \int_0^\infty dk k^{\nu-1} \exp\left\{-\frac{x}{2} \left(k + \frac{1}{k}\right)\right\}. \quad (3.2)$$

⁷ For a detailed treatment of Gould and DeWitt in the context of BPS, see Appendix B of Ref. [1].

As the notation in (3.2) suggests, we can think of ν as the dimension of space and the integration variable k as the wave number. In this analogy, the argument x corresponds to the dimensionless coupling parameter g of the plasma. The following dictionary provides a useful mnemonic in relating this mathematical example to the plasma physics problem of real interest:

$$\begin{aligned} x &\rightarrow \text{plasma coupling } g \\ \nu &\rightarrow \text{spatial dimension} \\ k &\rightarrow \text{wave number} \\ dk k^{\nu-1} &\rightarrow \text{integration measure } d^\nu k . \end{aligned} \tag{3.3}$$

Pushing our physical analogy further, if we think of ν as being the dimension of space, then it should always be a positive integer, which I will express by the conventional set theory notation $\nu \in \mathbb{Z}^+$; however, nothing *per se* in the integral representation (3.2) requires that $\nu \in \mathbb{Z}^+$. We can therefore think of ν in expression (3.2) as being a continuous real variable ($\nu \in \mathbb{R}$), or indeed, a complex variable ($\nu \in \mathbb{C}$) if circumstances warrant. Similarly, for a real physical system written in the appropriate integral form, there is nothing in any law of physics that prevents us from interpreting the dimension of space ν as being a complex number. Continuing ν from the positive integers into the complex plane is what I mean by “dimensional continuation.” There will be times when we restrict our attention to the real numbers only, rather than the complex numbers in general, and I will refer to this as dimensional continuation as well. As we shall see, this procedure of taking $\nu \in \mathbb{R}$ or $\nu \in \mathbb{C}$ will allow us to regulate otherwise infinite integrals in a systematic and perturbative fashion. Finite manipulations can then be performed, the divergent poles will cancel from physically measurable quantities, and afterward we can take ν to the appropriate integer dimension (in this analogy we take $\nu \rightarrow 0$, rather than $\nu \rightarrow 3$ as we do for the physics problem).

B. Leading Order Terms

Let us first calculate the leading order in x behavior of $K_\nu(x)$ for positive and negative values of ν . For small positive values of x , the leading order x -behavior can be obtained by replacing the exponential in (3.2) by one, *except* in the regions $k \rightarrow 0$ and $k \rightarrow \infty$, where the exponential is required for convergence. Taking $\nu < 0$ first, note that the integral (3.2) is dominated by small values of k near the lower limit of integration. In terms of the analogy (3.3) where k is a wave number, this corresponds to the situation in which long-distance IR physics is dominant. Therefore, when $\nu < 0$ and the integral is dominated by small values of

k , the leading order contribution to (3.2) can be obtained from the leading order behavior of the exponential, that is to say, the replacement

$$\exp \left\{ -\frac{x}{2} \left(k + \frac{1}{k} \right) \right\} \rightarrow \exp \left\{ -\frac{x}{2k} \right\} \quad \text{with } |x| \ll 1, \quad \nu < 0 \quad (3.4)$$

will capture the entire leading order in x behavior for negative values of ν . Note that (3.4) provides convergence as $k \rightarrow 0$, while large- k convergence is provided by the prefactor $k^{\nu-1}$ since $\nu < 0$. We will denote this leading order contribution by $K_\nu^<(x)$, and using the substitution (3.4) we write

$$K_\nu^<(x) = \frac{1}{2} \int_0^\infty dk k^{\nu-1} e^{-x/2k} = \frac{1}{2} \left(\frac{x}{2} \right)^\nu \Gamma(-\nu) . \quad (3.5)$$

In the last equality of (3.5), we have made the variable change $y = x/2k$, and we have used the standard integral representation for the Gamma function,

$$\Gamma(z) = \int_0^\infty dy y^{z-1} e^{-y} . \quad (3.6)$$

This representation of the Gamma function only converges when $z > 0$ (actually, (3.6) is valid over the complex z -plane with $\text{Re } z > 0$). When $z = -\nu$ and $\nu < 0$, we may indeed use (3.6).

We can find the leading order in x contribution when $\nu > 0$ in a similar manner. In this case, the integral is dominated by large values of k at the upper limit of integration, and we make the substitution

$$\exp \left\{ -\frac{x}{2} \left(k + \frac{1}{k} \right) \right\} \rightarrow \exp \left\{ -\frac{xk}{2} \right\} \quad \text{with } |x| \ll 1, \quad \nu > 0, \quad (3.7)$$

thereby giving the leading order result

$$K_\nu^>(x) = \frac{1}{2} \int_0^\infty dk k^{\nu-1} e^{-xk/2} = \frac{1}{2} \left(\frac{x}{2} \right)^{-\nu} \Gamma(\nu) . \quad (3.8)$$

Note that the exponential provides convergence as $k \rightarrow \infty$, while the integrand possesses an integrable singularity at $k = 0$ for $0 < \nu < 1$ (the integrand is non-singular at $k = 0$ when $\nu \geq 1$).

We will eventually take the $\nu \rightarrow 0$ limits of (3.5) and (3.8), since we are interested in $K_0(x)$ and not $K_\nu(x)$. While we could take ν to be a general real or complex number until the limit is required, it is easier to consider only small values of ν from the start (we need work no higher than linear order, since this and higher orders vanish when $\nu \rightarrow 0$).

To do this, we expand the Gamma function $\Gamma(\pm\nu)$ to linear-order in its argument using $\Gamma(z) = 1/z - \gamma + \mathcal{O}(z)$. Taking $z = \pm\nu$ in this expansion gives

$$K_\nu^>(x) = \frac{1}{2\nu} \left(\frac{x}{2}\right)^{-\nu} [1 - \nu\gamma] \quad : \text{ LO in } x \text{ when } \nu > 0 \quad (3.9)$$

$$K_\nu^<(x) = -\frac{1}{2\nu} \left(\frac{x}{2}\right)^\nu [1 + \nu\gamma] \quad : \text{ LO in } x \text{ when } \nu < 0 . \quad (3.10)$$

Expressions (3.9) and (3.10) are accurate to linear order in the dimension ν ; on the other hand, (3.9) gives the leading order in x contribution to $K_\nu(x)$ as defined by (3.2) when $\nu > 0$, and (3.10) gives the leading order in x contribution to $K_\nu(x)$ when $\nu < 0$. To compare these two expressions, we must analytically continue them to a common dimension. We will discuss this further in the next section.

In terms of our physics analogy, expression (3.9) captures the leading order short-distance physics in the $\nu > 0$ regime; the pole at $\nu = 0$ corresponds to a small- k divergence, which, pushing our physical analogy again, would reflect missing or incomplete long-distance physics (as with the Boltzmann equation). The situation is completely reversed for (3.10), which captures the leading order long-distance physics for $\nu > 0$, with the pole at $\nu = 0$ corresponding to a large- k divergence arising from missing short-distance physics (like the Lenard-Balescu equation). As functions of ν , we see from (3.9) and (3.10) that $K_\nu^>(x)$ and $K_\nu^<(x)$ are analytic in ν , except for the simple pole at $\nu = 0$. As we shall see, the analytic continuation to complex ν takes the same functional form as the individual expressions (3.9) and (3.10), each defined separately for $\nu > 0$ and $\nu < 0$, respectively.

C. Some Comments on Analytic Continuation

Since analytic continuation plays such a central role in dimensional continuation, at least mathematically, I would like to briefly discuss the conditions under which a function can be analytically continued from one region of the complex plane to another. Recall that a function f is said to be *analytic at a point* z_0 in the complex plane \mathbb{C} , if and only if its derivative exists not only at z_0 , but also at every point within some open neighborhood of z_0 . A function f is *analytic on a domain* D in the complex plane if it is analytic at each point in D . Analyticity is a very stringent condition on a function, since the existence of the derivative of a complex function is a much more robust constraint than the corresponding existence of the derivative of a function on a real domain. This is because in the two-dimensional complex plane, the limiting procedure defining the derivative must exist regardless of the direction used in taking the limit. In fact, analyticity at a point z_0 is so strong that it implies the existence and continuity of all derivatives $f^{(n)}(z_0)$ for any order $n > 0$ [11]. In other

words, an analytic function on D can be thought of as being infinitely smooth on D , even though the definition of analyticity itself invokes only the existence of the first derivative, albeit on a neighborhood.

Analyticity is such a stringent condition, that the behavior of an analytic function in a small domain is enough to determine its behavior in a larger region. Even if the function is only known along a one-dimensional curve in the complex plane, such as a portion of the real axis, this is enough to uniquely determine the function in the complex plane.⁸ This is because the derivatives of an analytic function exist to all orders, and therefore a Taylor series expansion about a point of analyticity always exists with some non-zero radius of convergence. For example, one of the powers of analytic continuation is that the original function may be defined in any manner over a subregion, and this can generate a unique function over a larger region. As an application of this, take the analytic function defined by the infinite geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n , \quad (3.11)$$

a series that converges only for $|z| < 1$. Upon defining f by (3.11), we therefore take the domain \mathbb{D}_1 to be the unit disk about the origin, excluding the unit circle itself. In this domain, the geometric series converges to

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } z \in \mathbb{D}_1 . \quad (3.12)$$

Note, however, that the function $g(z) = 1/(1-z)$ is defined over the entire complex plane except $z = 1$, a region I will call \mathbb{C}_1 . Since the function f is only defined within the unit circle, and since f and g agree within the unit circle, the function $g: \mathbb{C}_1 \rightarrow \mathbb{C}$ is the unique analytic continuation of $f: \mathbb{D}_1 \rightarrow \mathbb{C}$.

As a more relevant example, consider $K_{\nu}^{<}(x)$ with $\nu < 0$ as given by (3.10). To compare this with $K_{\nu}^{>}(x)$, which is determined by (3.9) for $\nu > 0$, we must analytically continue $K_{\nu}^{<}(x)$ to the positive real ν -axis. We can think of $K_{\nu}^{<}(x)$ as a sequence of functions of an independent variable ν indexed by a continuous label x ; therefore, in a more suggestive notation, we temporarily write $f_x(\nu) \equiv K_{\nu}^{<}(x)$. While the collection of functions $f_x(\nu)$ are defined in (3.10) on the negative ν -axis \mathbb{R}^- (excluding the simple pole at zero), they can be

⁸ In fact, Carlson's Theorem [12] can be used to uniquely extend a function defined only on the integers to the whole complex plane. This is actually the theorem of most relevance here; however, as we shall see in Section IV B, in practice we can analytically continue a function defined on the integers without this Theorem. We appeal to Carlson's Theorem only to guarantee the uniqueness of this procedure.

analytically continued to the complex ν -plane \mathbb{C} . Furthermore, the functions take the same algebraic form on the complex plane, namely,

$$f_x(\nu) = -\frac{1}{2\nu} \left(\frac{x}{2}\right)^\nu \left[1 + \nu\gamma\right] \quad \text{for } \nu \in \mathbb{C}. \quad (3.13)$$

We can now restrict our attention from \mathbb{C} in general to the positive ν -axis \mathbb{R}^+ (excluding zero). This allows us to directly compare $K_\nu^<(x)$ and $K_\nu^>(x)$ at $\nu \in \mathbb{R}^+$ using the same algebraic forms as given by (3.10) and (3.9). In the next section, we discuss the implications of analytically continuing $K_\nu^<(x)$ from the negative axis $\nu < 0$ to the positive axis $\nu > 0$. Alternately, we could continue $g_x(\nu) \equiv K_\nu^>(x)$ to the region $\nu \in \mathbb{R}^-$ using the same functional form as (3.9), and compare this with $K_\nu^<(x)$.

D. Next-to-Leading Order Term

We now illustrate the key mathematical result that allows dimensional continuation to extract not only the leading, but the next-to-leading order terms. Recall from (3.9) and (3.10) that $K_\nu^>(x)$ and $K_\nu^<(x)$ are both leading order in x for $\nu > 0$ and $\nu < 0$, respectively. Since these functions were calculated for mutually exclusive values of ν , one might think that they cannot be compared. When viewed as an analytic function in the complex ν -plane, however, we have seen that $K_\nu^<(x)$ is also a function over the domain $\nu > 0$, in which case both $K_\nu^<(x)$ and $K_\nu^>(x)$ can be compared at the same values of ν and x . Since the algebraic form is so simple, $K_\nu^<(x)$ takes the same functional form when analytically continued to $\nu > 0$ as it did for $\nu < 0$. As illustrated in Fig. 1, this means that $K_\nu^<(x)$ becomes next-to-leading order in x along the positive real axis:

$$K_\nu^>(x) = \frac{1}{2\nu} \left(\frac{x}{2}\right)^{-\nu} \left[1 - \nu\gamma\right] \quad : \text{ LO in } x \text{ when } \nu > 0 \quad (3.14)$$

$$K_\nu^<(x) = -\frac{1}{2\nu} \left(\frac{x}{2}\right)^\nu \left[1 + \nu\gamma\right] \quad : \text{ NLO in } x \text{ when } \nu > 0. \quad (3.15)$$

The expressions (3.14) and (3.15) remain accurate to linear-order in ν .

To see that (3.15) is indeed next-to-leading order in x relative to (3.14), note that the x -behavior of the leading order contribution for $\nu > 0$ can be written $K_\nu^>(x) \sim x^{-|\nu|}$. I have used the absolute value $|\nu|$ to emphasize that the power of x in (3.14) is strictly negative when $\nu > 0$. Similarly, along the positive ν -axis we find the behavior $K_\nu^<(x) \sim x^{|\nu|}$ for $\nu > 0$, and we see that $x^{-|\nu|} \gg x^{|\nu|}$ for $0 < x \ll 1$. This means $K_\nu^>(x) \gg K_\nu^<(x)$ for $\nu > 0$ and $0 < x \ll 1$, and we are therefore justified in calling $K_\nu^>(x)$ leading order in x and $K_\nu^<(x)$ next-to-leading order.

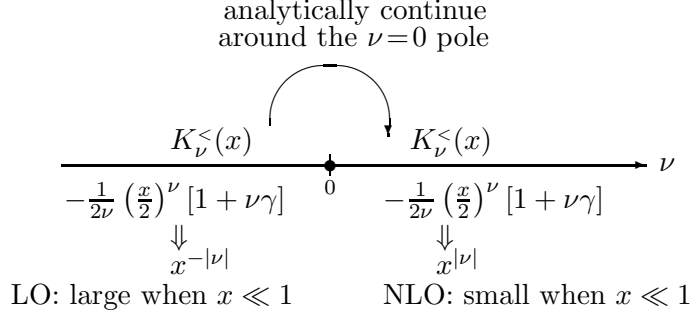


FIG. 1: The analytic continuation of $K_\nu^<(x)$ from $\nu \in \mathbb{R}^-$ to $\nu \in \mathbb{R}^+$ in the complex ν -plane: the same expression can be used for $K_\nu^<(x)$ throughout the complex plane since the pole at $\nu = 0$ can easily be avoided, as indicated in the figure. Note that $K_\nu^<(x) \sim x^{-|\nu|}$ is leading order in x for $\nu < 0$. However, upon analytically continuing $K_\nu^<(x)$ to $\nu > 0$, the x -dependence becomes $K_\nu^<(x) \sim x^{|\nu|}$, and the function is next-to-leading order relative to $K_\nu^>(x) \sim x^{-|\nu|}$.

Strictly speaking, we have only shown that $K_\nu^<(x)$ is subleading relative to $K_\nu^>(x)$ when $\nu > 0$. To conclude that $K_\nu^<(x) \sim x^\nu$ is indeed next-to-leading order relative to $K_\nu^>(x) \sim x^{-\nu}$, it is important to establish that there are no powers of x between x^ν and $x^{-\nu}$ in the expansion of $K_\nu(x)$. For $\nu > 0$, one simply subtracts (3.9) from (3.2), and it becomes clear that this error is higher order in x than x^ν . For a more detailed proof of this, see footnote 2 of Ref. [8]. A similar statement holds for $K_\nu^>(x)$, namely, as we analytically continue from $\nu > 0$ to $\nu < 0$, the quantity $K_\nu^>(x)$ switches from leading order to next-to-leading order in x relative to $K_\nu^<(x)$.

E. Assembling the Pieces

We have now assembled enough results to find $K_0(x)$ to leading and next-to-leading order in x : we simply add the expressions (3.14) and (3.15) and take the limit of vanishing ν . Note that this does not lead to any form of “double counting.” Instead, we are simply adding the next-to-leading order term (3.15) to the leading order term (3.14) at a common value of $\nu > 0$. Upon taking the limit of vanishing ν , or more precisely $\nu \rightarrow 0^+$ since ν is always positive in (3.14) and (3.15), we obtain $K_0(x)$ to leading and next-to-leading order in x .

We now calculate this limit, proving that

$$\lim_{\nu \rightarrow 0^+} [K_\nu^>(x) + K_\nu^<(x)] = -\ln x + \ln 2 - \gamma. \quad (3.16)$$

Let us first expand $(x/2)^{\pm\nu}$ in powers of ν . We will denote $z = x/2$, from which we find $z^{\pm\nu} = e^{\ln(z^{\pm\nu})} = e^{\pm\nu \ln z} = 1 \pm \nu \ln z + \mathcal{O}(\nu^2)$, or in summary:

$$z^{\pm\nu} = 1 \pm \nu \ln z + \mathcal{O}(\nu^2). \quad (3.17)$$

This allows us to express (3.14) and (3.15) as

$$K_\nu^>(x) = \frac{1}{2\nu} \left[1 - \nu \ln\left(\frac{x}{2}\right) \right] \left[1 - \nu\gamma \right] + \mathcal{O}(\nu) = \frac{1}{2\nu} - \frac{1}{2} \ln\left(\frac{x}{2}\right) - \frac{\gamma}{2} + \mathcal{O}(\nu) , \quad (3.18)$$

and

$$K_\nu^<(x) = -\frac{1}{2\nu} \left[1 + \nu \ln\left(\frac{x}{2}\right) \right] \left[1 + \nu\gamma \right] + \mathcal{O}(\nu) = -\frac{1}{2\nu} - \frac{1}{2} \ln\left(\frac{x}{2}\right) - \frac{\gamma}{2} + \mathcal{O}(\nu). \quad (3.19)$$

When we divide (3.17) by a factor of ν , as required by (3.14) and (3.15), note that we find: (i) a pole $1/\nu$ from the first term in (3.17), (ii) a non-analytic finite contribution $\pm \ln z$ from the second term, and (iii) the error in ν becomes $\mathcal{O}(\nu)$, which is the same order in ν that we are neglecting in (3.14) and (3.15). The error in ν , which vanishes in the limit $\nu \rightarrow 0^+$, should not be confused with the error in x , the latter being $\mathcal{O}(x^2)$ for vanishing ν . Note that the pole terms cancel upon adding (3.18) and (3.19), so that

$$K_\nu^>(x) + K_\nu^<(x) = -\ln\left(\frac{x}{2}\right) - \gamma + \mathcal{O}(\nu) , \quad (3.20)$$

thereby giving (3.16) as $\nu \rightarrow 0^+$. As we have discussed, there are no x -dependent terms that lie between $K_\nu^>(x) \sim x^{-\nu}$ and $K_\nu^<(x) \sim x^\nu$, so this procedure has captured the leading and next-to-leading order behavior in x .

In exactly the same way, we can also calculate the leading order and next-to-leading order contribution to $K_0(x)$ by taking the limit from the left,

$$\lim_{\nu \rightarrow 0^-} \left[K_\nu^<(x) + K_\nu^>(x) \right] = -\ln x + \ln 2 - \gamma . \quad (3.21)$$

I should point out a potential notational problem in (3.16) and (3.21). Concentrating on (3.16) for the moment, the limit $\nu \rightarrow 0^+$ indicates that both terms $K_\nu^>$ and $K_\nu^<$ are understood to live in dimensions $\nu > 0$, with the second term $K_\nu^<$ having been analytically continued from $\nu < 0$. The notation with which the term $K_\nu^<$ is written in (3.16), however, does not indicate that it has been analytically continued. This should be no cause for confusion, however, since $K_\nu^<$ takes the same functional form in any dimension ν ; therefore, a separate notation indicating that the $K_\nu^<$ in (3.16) has been analytically continued is unnecessary. We can simply add $K_\nu^>$ and $K_\nu^<$ as calculated in $\nu > 0$ and $\nu < 0$ respectively.

IV. DIMENSIONAL CONTINUATION

A. Rate of Energy Exchange as a Perturbative Expansion

1. The g -Expansion

Before moving on to the details of dimensional continuation, we must first discuss the plasma expansion parameter g . Since the problems in plasma physics are usually so complicated as to preclude a perturbative approach, most plasma physicists do not usually think in terms of expanding systematically in a small dimensionless parameter. However, for a weakly to moderately coupled plasma, it is a quite fruitful approach to perturbatively expand the rate in a small dimensionless coupling constant.

That such a universal parameter for a plasma exists was discussed at length in Ref. [13], where it was shown that any physical quantity associated with a plasma whose species are in equilibrium with themselves (such as the plasma we are studying) can be expanded in *integer* powers of a dimensionless coupling constant g defined by

$$g \equiv \frac{\text{Coulomb Energy for Two Charges at Separation } \lambda_D}{\text{Temperature in Energy Units}}, \quad (4.1)$$

with λ_D being the Debye length of the plasma. Since the potential energy between two like charges is given by $V = e^2/4\pi r$ in rationalized units, and writing the Debye wave number as $\kappa_D = \lambda_D^{-1}$, the coupling parameter is therefore

$$g = \frac{e^2 \kappa_D}{4\pi T}. \quad (4.2)$$

For a multicomponent plasma, there is actually a coupling constant for each pair of components,

$$g_{ab} = \frac{e_a e_b \kappa_b}{4\pi T_b}, \quad (4.3)$$

with κ_b defined by (1.4). However, when the pairs have approximately the same coupling strength, then the single parameter (4.2) adequately characterizes the entire plasma. Expressing the charges as $e_a = Z_a e$ and $e_b = Z_b e$, we can write g_{ab} as

$$g_{ab} = Z_a Z_b^2 \frac{e^3}{4\pi} \frac{n_b^{1/2}}{T_b^{3/2}}, \quad (4.4)$$

and we see that the coupling constant is proportional to the cube of the electric charge, the square root of the density, and the inverse (3/2)-power of the temperature.

Recall that the usual plasma parameter Γ is defined in a similar manner to (4.1), except that the charge separation is determined not by λ_D , but by the inter-particle spacing in the plasma,

$$\Gamma \equiv \frac{\text{Coulomb Energy for Two Charges at Inter-particle Separation } d}{\text{Temperature in Energy Units}} . \quad (4.5)$$

The inter-particle spacing d is defined in several ways in the literature, but the idea is to transform the plasma number density n into a length scale, so that $d \propto n^{-1/3}$. The most common convention is to define d to be the radius of a sphere containing, on average, a single plasma particle, so that $4\pi d^3/3 = 1/n$, and therefore

$$\Gamma = \frac{e^2}{4\pi T} \left(\frac{4\pi n}{3} \right)^{1/3} . \quad (4.6)$$

With this convention, the relation between the two plasma coupling parameters for a single plasma species is $\Gamma^3 = g^2/3$, and for an arbitrary number of plasma species we always find $g \propto \Gamma^{3/2}$. We can therefore use either g or Γ to characterize the strength of the plasma, as g and Γ become large or small together.⁹ For our purposes, however, there is an advantage to using g rather than Γ since physical quantities are expanded as *integer* powers of g , while they expand in fractional powers of Γ .

2. Next-to-Leading Order and the Coulomb Logarithm

As I have said, any plasma quantity can be written as a power series expansion in integer power of g , with the possible exception of non-analytic terms involving $\ln g$. For the process of energy exchange via Coulomb interactions, this non-analyticity arises from the competition between disparate physical length scales. As an expansion in g , the rate of energy exchange takes the form

$$\frac{d\mathcal{E}}{dt} = - \underbrace{A g^2 \ln g}_{\text{LO}} + \underbrace{B g^2}_{\text{NLO}} + \mathcal{O}(g^3) . \quad (4.7)$$

⁹ The dimensional continuation method requires that we perform calculations in an arbitrary spatial dimension ν ; however, the parameter $g = e^2 \kappa_D / 4\pi T$ as given by (4.2) is *dimensionless* (in the engineering sense) only in three spatial dimensions. As such, it is meaningless to call g large or small in any dimension other than $\nu=3$. This is no cause for alarm, however, since in footnote 21, I will construct a dimensionless coupling parameter g_ν in arbitrary spatial dimensions ν . The parameter g_ν can then be used to characterize the plasma strength in ν dimensions, and it will have the properties that (i) $g_\nu \propto g$ and (ii) $g_\nu \rightarrow g$ as $\nu \rightarrow 3$. Property (i) implies that a g_ν -expansion is the same as a g -expansion, and (ii) implies this correspondence continues down to three dimensions, where g becomes the relevant expansion parameter.

In (4.7), I have indicated the leading order in g (LO) and the next-to-leading order in g (NLO) terms in the g -expansion: the first term is leading order relative to the second because $|g^2 \ln g| > g^2$ for small g . The minus sign on the leading order term of (4.7) is a matter of convention. Since the logarithm $\ln g$ will be negative in a weakly coupled plasma (recall $g < 1$), the minus sign renders the coefficient A positive when the energy exchange is positive. The coefficient A was first calculated by Spitzer. The coefficient B , however, is very difficult to calculate, and this was the main purpose of BPS [1]. It is convenient to define the dimensionless coefficient C by $B = -A \ln C$, in which case we can write

$$\frac{d\mathcal{E}}{dt} = A g^2 \ln \Lambda_{\text{coul}} + \mathcal{O}(g^3) , \quad \text{with} \quad \ln \Lambda_{\text{coul}} = -\ln\{Cg\} . \quad (4.8)$$

We see, then, that knowing the next-to-leading order term is equivalent to knowing the exact coefficient C under the logarithm. Note that the minus sign renders the Coulomb logarithm positive when g is very small, in keeping with convention.

3. Factors of g Inside the Coulomb Logarithm

For the heuristic model building of Section II A, let us pause for a moment and show that the argument of the Coulomb logarithm in (2.1) is indeed proportional to g , as required by (4.8). On physical grounds we saw that the long-distance scale b_{max} is set by a Debye length, and therefore we nominally set $b_{\text{max}} = \kappa_{\text{D}}^{-1}$. In the extreme classical limit, the short-distance cutoff b_{min} is set by the classical distance of closest approach r_{cl} , so that $b_{\text{min}} = c r_{\text{cl}}$. For simplicity, we will choose the coefficient c such that $K = A g^2$ between (2.1) and (4.8), in which case

$$\ln \Lambda_{\text{coul}} = \ln \left\{ \frac{b_{\text{max}}}{b_{\text{min}}} \right\} . \quad (4.9)$$

Let us consider two unit charges of mass m approaching one another with zero impact parameter. The rms speed of each particle is determined by

$$\frac{1}{2} m \bar{v}^2 = \frac{3}{2} T \quad \Rightarrow \quad \bar{v} = \sqrt{\frac{3T}{m}} , \quad (4.10)$$

while energy conservation $\frac{1}{2} m \bar{v}^2 + \frac{1}{2} m \bar{v}^2 = e^2 / 4\pi r_{\text{cl}}$ gives the distance of closest approach,

$$r_{\text{cl}} = \frac{e^2}{4\pi m \bar{v}^2} = \frac{e^2}{4\pi} \frac{1}{3T} . \quad (4.11)$$

In the extreme classical regime, we see that the argument of the Coulomb logarithm in (4.9) is indeed proportional to the plasma coupling constant,

$$\frac{b_{\text{min}}}{b_{\text{max}}} = c r_{\text{cl}} \kappa_{\text{D}} = \frac{c}{3} \frac{e^2 \kappa_{\text{D}}}{4\pi T} = \frac{c}{3} g . \quad (4.12)$$

Let us now look at the *ad hoc* interpolation (2.2) between the classical and quantum regimes. Up to this point I have said very little about quantum mechanics. While I will not dwell on quantum corrections, I will briefly discuss a dimensionless expansion parameter that characterizes the strength of the quantum two-body scattering correction. There are many ways of defining such a parameter, but I will follow Ref. [13], taking

$$\eta \equiv \frac{\text{Classical Distance of Closest Approach}}{\text{Thermal Wavelength}} . \quad (4.13)$$

With this definition, quantum corrections are large when $\eta \ll 1$. Motivated by the de Broglie wavelength of a particle, the thermal wavelength of a plasma species is given by $r_{\text{qm}} = \hbar/\bar{q}$, where $\bar{q} = m\bar{v}$ is a typical momentum transfer suffered during a collision. Definition (4.13) yields $\eta = r_{\text{cl}}/r_{\text{qm}} = (e^2/4\pi m\bar{v}^2) \cdot (m\bar{v}/\hbar)$, or more succinctly

$$\eta = \frac{e^2}{4\pi\hbar\bar{v}} , \quad (4.14)$$

from which (2.2) gives

$$\frac{b_{\text{min}}}{b_{\text{max}}} = \frac{c}{3} \left(1 + \frac{1}{\eta^2}\right)^{1/2} g . \quad (4.15)$$

In the extreme quantum limit in which $\eta \ll 1$, this becomes $b_{\text{min}}/b_{\text{max}} = (c/3) (g/\eta)$.

Finally, note that the factors inside the BPS Coulomb logarithm (1.3) are also proportional to the coupling constant g , and upon dropping the electron subscripts for convenience, we find

$$\frac{\hbar^2\omega^2}{T^2} = \frac{1}{3} \frac{g^2}{\eta^2} . \quad (4.16)$$

We see that the g -dependence of the Coulomb logarithm arises quite naturally. However, the accompanying coefficient under the logarithm might also possess η -dependence, thereby obscuring the g -dependence unless we are careful.

B. Mathematics of Dimensional Continuation

Before describing what dimensional continuation is, allow me to first state what it is not. Dimensional continuation is *not* performing an integral to a fractional power of the spatial dimension, as with the meaningless expression

$$\int_0^\infty d^{\frac{3}{2}}k f(k) .$$

Instead, dimensional continuation is the following. Suppose some physical quantity of interest can be written as an integral over a kernel

$$Q(m) = \int_{\mathbb{R}^3} d^3k f(\mathbf{k}; m) , \quad (4.17)$$

where f is determined by the physical equations of motion, whether classical or quantum. The integrand is of course a function of the physical parameters, such as the masses and charges of the fundamental particles, and I have abbreviated this dependence by the parameter m . For definiteness, we will think of k as a wave number with dimensions of an inverse length. The laws of physics, from which (4.17) follows, are usually written in three dimensional space. Thus, we usually take \mathbf{k} to be a three-dimensional vector, and we integrate \mathbf{k} over the entire three-dimensional Euclidean space \mathbb{R}^3 .

The known fundamental laws of physics themselves, however, do not specify a particular spatial dimension in which they hold. In fact, as far as the known laws of physics are concerned, the actual value ν of the spatial dimension can be viewed as a free *integer* parameter: it is simply an unexplained empirical fact that we live in three dimensions.¹⁰ We can therefore express any three-dimensional physical quantity or law, such as Newton's equation of motion or Gauss' law, in any number of integer dimensions $\nu \in \mathbb{Z}^+$. For example, we can write down a corresponding quantity to (4.17) in an arbitrary number of dimensions,

$$Q(\nu; m) = \int_{\mathbb{R}^\nu} d^\nu k f_\nu(\mathbf{k}; m) , \quad (4.18)$$

where the wave vector \mathbf{k} is now a ν -dimensional vector, and we integrate over the entire ν -dimensional Euclidean space \mathbb{R}^ν . I have placed a subscript on the integrand f_ν to indicate that it is determined by the theory expressed in ν dimensions. At this point, the spatial dimension is a non-negative *integer*, so that $\nu = 1, 2, 3, 4, \dots$. Since the integer ν is arbitrary in (4.18), I have indicated that the corresponding quantity contains ν -dependence by writing $Q(\nu; m)$; however, for notational simplicity I will drop the parametric dependence of quantities such as mass and simply write $Q(\nu)$. A quantity that diverges in three-dimensions will be finite in arbitrary ν , but it will typically exhibit a simple pole of the form

$$Q(\nu) = \frac{Q_0}{\nu - 3} + Q_1(\nu) , \quad (4.19)$$

¹⁰ One would expect that the fundamental theory of nature, a theory of everything, would predict the number of space-time dimensions, solving this mystery at last. One of the great successes of string theory is that it is one of only two known theories that indeed predicts the number of space-time dimensions – the theory is inconsistent in all but nine space and one time dimensions. Accordingly, this is also one of the great failures of string theory. The other theory is super-gravity, which is only consistent in eleven dimensions.

where $Q_1(\nu)$ is finite at $\nu = 3$.¹¹ Dimensional continuation is simply the act of treating $Q(\nu)$ as a function of a complex argument ν , *after* the integral (4.18) has been performed for *all* positive integer values of ν . This is not an unfamiliar procedure, as the factorial function $\nu!$ with $\nu \in \mathbb{Z}^+$ can be generalized to the gamma function $\Gamma(\nu + 1)$ in which $\nu \in \mathbb{C}$. Indeed, given any function $Q(\nu)$ defined on the integers, with only mild restrictions placed on the function at large values of the argument, Carlson's Theorem [12] allows us to continue this function uniquely to the complex plane.

How does dimensional continuation work in practice? We will look at a few specific examples in Sections IV C and VI, but for now let us consider a general physical quantity Q in which the integrand in (4.17) depends solely upon the modulus of \mathbf{k} , so that

$$Q = \int d^3k f(k) . \quad (4.20)$$

In such a case it is not uncommon that the integrand in the generalization (4.18) is only a function of the modulus of the ν -dimensional wave vector \mathbf{k} , with the same functional form as the integrand in (4.20). In other words, in (4.18) we have $f_\nu(\mathbf{k}) = f(k)$ with $k = |\mathbf{k}|$, thereby allowing us to write

$$Q(\nu) = \int d^\nu k f(k) . \quad (4.21)$$

Since the integrand is a function only of k , we can extract the angular integrals and write

$$Q(\nu) = \underbrace{\int d\Omega_{\nu-1}}_{(\nu-1) \text{ integrals}} \cdot \underbrace{\int_0^\infty k^{\nu-1} dk f(k)}_{\text{one-dimensional integral}} . \quad (4.22)$$

At this point, the dimension ν is simply an arbitrary positive integer, $\nu \in \mathbb{Z}^+$. As we will show in the next paragraph, the integration over all angles gives

$$\Omega_{\nu-1} \equiv \int d\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} . \quad (4.23)$$

This leaves a one-dimensional integral to perform, in which ν simply acts as a parameter,

$$F(\nu) \equiv \int_0^\infty dk k^{\nu-1} f(k) . \quad (4.24)$$

The physical quantity now becomes the product of (4.23) and (4.24) with $\nu \in \mathbb{Z}^+$. In the case of (4.23), we already know how to analytically continue ν to complex values. On the other

¹¹ As we shall see in the next section, it is the nature of the Coulomb force that gives a pole in three dimensions, rather than some other value of the dimension. We shall further see that there is important physics in the residue Q_0 of the pole.

hand, for (4.24) we can think of ν as being an arbitrary complex number when performing the one-dimensional integral over k , and therefore we can regard $F(\nu)$ as a function over the complex ν -plane, thereby giving

$$Q(\nu) = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} F(\nu) \quad \text{with } \nu \in \mathbb{C} . \quad (4.25)$$

In this manner, we can regard $Q(\nu)$ as a function of a complex argument ν , and by Carlson's Theorem [12], this is the unique continuation from positive integer values of ν to complex values of ν .

As an example of this procedure, let us prove (4.23). First, consider the one-dimensional Gaussian integral

$$\int_{-\infty}^{\infty} dk e^{-k^2} = \sqrt{\pi} . \quad (4.26)$$

If we multiply both sides together ν times (with $\nu \in \mathbb{Z}^+$), we find

$$(\sqrt{\pi})^\nu = \int_{-\infty}^{\infty} dk_1 e^{-k_1^2} \int_{-\infty}^{\infty} dk_2 e^{-k_2^2} \dots \int_{-\infty}^{\infty} dk_\nu e^{-k_\nu^2} = \int d^\nu k e^{-\mathbf{k}^2} , \quad (4.27)$$

where the wave vector \mathbf{k} in the exponential of the last expression is the ν -dimensional vector $\mathbf{k} = (k_1, k_2, \dots, k_\nu)$, and \mathbf{k}^2 is the ν -dimensional inner product $\mathbf{k}^2 = \sum_{\ell=1}^{\nu} k_\ell^2$. As in (4.22), we can factor the angular integrals out of the right-hand-side of (4.27), and the remaining one-dimensional integral can be converted to a Gamma function with the change of variables $t = k^2$:

$$\pi^{\nu/2} = \int d\Omega_{\nu-1} \cdot \int_0^\infty dk k^{\nu-1} e^{-k^2} = \int d\Omega_{\nu-1} \cdot \frac{1}{2} \int_0^\infty dt t^{\nu/2-1} e^{-t} = \int d\Omega_{\nu-1} \cdot \frac{1}{2} \Gamma(\nu/2) . \quad (4.28)$$

Solving for $\int d\Omega_{\nu-1}$ in (4.28) gives (4.23). As an aside, it is interesting to note that we have also found the hyper-area of a $(\nu-1)$ -dimensional sphere of radius r in \mathbb{R}^ν ,

$$A_\nu = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} r^{\nu-1} , \quad (4.29)$$

and integrating (4.29) gives the hyper-volume of a ν -dimensional ball of radius r ,

$$\Sigma_\nu = \frac{\pi^{\nu/2}}{\Gamma(\nu/2 + 1)} r^\nu . \quad (4.30)$$

C. Physics of Dimensional Continuation

So far we have only introduced dimensional continuation as a regularization prescription, a means by which infinite theories can be rendered temporarily finite. This is the use to which dimensional continuation is commonly employed in field theory; however, this alone is not sufficient to render the method suitable for our purposes. Instead, there are also physics reasons that make dimensional continuation applicable to the problem at hand. I will now show that the Coulomb potential, and indeed physics in general, behaves differently in different spatial dimensions. In particular, short-distance or ultraviolet (UV) physics dominates when $\nu > 3$; while long-distance or infrared (IR) physics dominates when $\nu < 3$. In $\nu = 3$ dimensions, both UV and IR physics are equally important.¹² This means that dimensional continuation acts as a “physics sieve,” allowing us to capture the leading UV and IR physics simply by performing the relevant integrals in dimensions greater than or less than the traditional $\nu = 3$. In the next few paragraphs, we will discuss why changing the dimension of space emphasizes either long- or short-distant physics. Understanding this point is crucial for all that follows.

We now turn to finding the ν -dimensional Coulomb potential, an exercise that succinctly illustrates how the physics of a system changes with the dimension of space. Besides illustrating that UV physics dominates in higher dimensions (and conversely), this example will also clarify the manner by which one performs physical calculations in arbitrary dimensions. Let us consider Poisson’s equation

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \rho(\mathbf{x}) . \quad (4.31)$$

There is nothing in this equation *per se* that restricts us to three dimensions.¹³ We *choose* to describe the electric field \mathbf{E} and the spatial coordinate \mathbf{x} as three-dimensional vectors because we live in three dimensions. However, from the mathematics alone, we could equally well consider these vectors as living in an arbitrary ν -dimensional space \mathbb{R}^ν . The rectilinear coordinates would then become $\mathbf{x} = (x_1, x_2, \dots, x_\nu)$, with a similar expression for the ν -dimensional electric field, while the gradient would be $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_\nu)$. It is convenient to write Poisson’s equation in its integral representation, which will allow us to calculate the electric field of a point charge with relative ease. Consider a point-charge at

¹² There are a number of such *coincidences*, both physical and mathematical, that suggest there is something special about living in three spatial dimensions and one time dimension. This could well be the Anthropic Principle at work.

¹³ One can most easily write the complete set of Maxwell’s equation in a general dimension by employing the Lorentz covariant form $\sum_{\alpha=0}^{\nu} \partial F^{\alpha\beta} / \partial x^\alpha = j^\beta$, where the electric and magnetic fields have been expressed in terms of the anti-symmetric field tensor $F^{\alpha\beta}$.

the origin given by $\rho(\mathbf{x}) = e \delta^{(\nu)}(\mathbf{x})$, and let Σ be any volume containing the charge e . Then, in a general number of dimensions, we can integrate (4.31) to obtain

$$\int_{\Sigma} d^{\nu}x \nabla \cdot \mathbf{E} = e . \quad (4.32)$$

Note that the dimensionality of space is now explicitly indicated by the integration measure.

For our purposes, the advantage of the integral representation of Poisson's equation is that the electric field of a point charge in the ν -dimensional space can easily be calculated by the same symmetry principles that hold in three dimensions. Suppose the volume Σ is a spherical ball B_r of radius r centered on the charge. The boundary of B_r is a sphere of dimension $\nu-1$ and will be denoted by ∂B_r (I am using the common notation ∂ in differential geometry for the boundary of a manifold). From rotational symmetry, the electric field \mathbf{E} of a point charge is directed radially outward and lies normal to the surface ∂B_r at each point. We will denote the magnitude of the electric field at radius r by $E(r)$. Recall that in (4.29), we calculated the hyper-area of the $(\nu-1)$ -sphere ∂B_r to be $A_{\nu} = \Omega_{\nu-1} r^{\nu-1}$ with $\Omega_{\nu-1} = \pi^{\nu/2}/\Gamma(\nu/2)$. Since the divergence theorem holds in an arbitrary Euclidean space (like the laws of physics, there is nothing in the divergence theorem to restrict the dimensionality of space to three), we can write

$$e = \int_{B_r} d^{\nu}x \nabla \cdot \mathbf{E} = \oint_{\partial B_r} d\mathbf{A} \cdot \mathbf{E} = \Omega_{\nu-1} r^{\nu-1} \cdot E(r) . \quad (4.33)$$

At position \mathbf{x} , the electric field therefore takes the form

$$\mathbf{E}(\mathbf{x}) = \frac{e}{\Omega_{\nu-1} r^{\nu-1}} \hat{\mathbf{x}} , \quad (4.34)$$

where I am using the notation $\mathbf{x} = r \hat{\mathbf{x}}$, with $\hat{\mathbf{x}}$ being a unit vector pointing in the direction of \mathbf{x} and $r = |\mathbf{x}|$ being the magnitude. It is often more convenient to work with the electric potential, a scalar quantity $\phi_{\nu}(r)$ defined by $E(r) = -d\phi_{\nu}(r)/dr$. In fact, we need the potential energy $V_{\nu}(r) = e \phi_{\nu}(r)$, so that

$$V_{\nu}(r) = \frac{1}{C_{\nu}} \frac{e^2}{r^{\nu-2}} , \quad (4.35)$$

with

$$C_{\nu} = \Omega_{\nu-1} (\nu - 2) = \frac{4\pi^{\nu/2}}{\Gamma(\nu/2 - 1)} . \quad (4.36)$$

For $\nu = 3$ we have $C_3 = 1/4\pi$, which is the origin of the 4π of rationalized units. Note from (4.35) that the engineering unit of electric charge is a function of the dimension ν . This is because the ν -dimensional Coulomb potential (4.35) must have units of energy, and

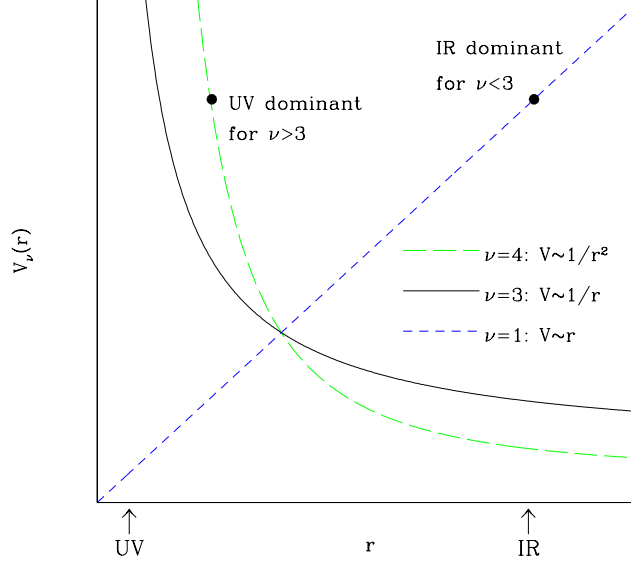


FIG. 2: Short-distance or ultraviolet (UV) physics dominates in dimensions $\nu > 3$. Long-distance or infrared (IR) physics dominates when $\nu < 3$. UV and IR physics are equally important in $\nu = 3$.

consequently the engineering unit of e^2 is $\text{Energy} \times (\text{Length})^{\nu-2} = \text{Mass} \times (\text{Length})^\nu / (\text{Time})^2$. It is quite natural that a composite quantity, made from the fundamental units of Mass, Length and Time, change its engineering dimension with the dimension of space.

Figure 2 shows the Coulomb potential for $\nu=3$, along with two representative dimensions on either side of $\nu = 3$. As the figure illustrates, the short-distance or UV behavior of the Coulomb potential becomes more severe as the dimension increases above $\nu = 3$, while the long-distance or IR behavior dominates for dimensions below $\nu = 3$. The arbitrary integration constant for the potential energy has been adjusted in each case so that all three graphs intersect at a single point. This was purely for aesthetics, as it renders the differences between the potentials more apparent. Despite the trouble we went through in the previous paragraph to find the coefficients C_ν , in this paragraph (and only in this paragraph) I have temporarily set $C_\nu = 1$. This will make it easier to compare the r -dependence of various potentials, and I would rather opt for clarity over notational consistency. For the representative potential with dimension below $\nu = 3$, I chose to graph the one-dimensional potential $V_1(r) = e^2 r$ rather than the two-dimensional potential $V_2(r) = e^2 \ln(r/r_0)$, where r_0 is an arbitrary integration constant with units of length. In both cases the potential increases without bound at long distances, thereby illustrating the dominance of long-distance or IR physics in dimensions less than three. However, I chose to graph $V_1(r)$ rather than $V_2(r)$ because the latter possesses a potentially misleading divergence as $r \rightarrow 0$: unlike the short-distance or UV divergence associated with the potential in dimensions greater than three, the

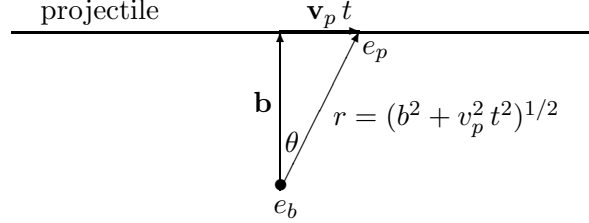


FIG. 3: A projectile of charge e_p , mass m_p , and velocity \mathbf{v}_p passing a fixed charge e_b . The impact parameter \mathbf{b} is normal to the velocity, so that $\mathbf{b} \cdot \mathbf{v}_p = 0$. The radial separation between the charges is $r(t) = (b^2 + v_p^2 t^2)^{1/2}$, and only the b -component $E_\perp = E \cos \theta = (e_b / \Omega_{\nu-1}) r^{-\nu}$ of the electric field contributes the impulse integration (4.40), where $\cos \theta = b/r$.

$r \rightarrow 0$ divergence of $V_2(r)$ is integrable, and consequently harmless. Therefore, for purposes of illustration, the potential $V_1(r) = e^2 r$ makes the point better than $V_2(r) = e^2 \ln(r/r_0)$. Remarkably, we now see that by simply selecting the dimension ν , we can dial a potential $V_\nu(r)$ that filters either long-distance or short-distance physics.

I would now like to show how the pole at $\nu = 3$ arises from the ν -dimensional Coulomb potential. In the temperature equilibration process, individual plasma species exchange energy by mutual Coulomb interactions. For example, consider a particle in the plasma with charge e_p , mass m_p , and velocity \mathbf{v}_p , and suppose it passes another charge e_b with an impact parameter \mathbf{b} . This is illustrated in Fig. 3, where, to zeroth order, the projectile follows the straight line $\mathbf{x}(t) = \mathbf{b} + \mathbf{v}_p t$ as a function of the time t , with $\mathbf{b} \cdot \mathbf{v}_p = 0$.¹⁴ In ν -dimensional space, the Coulomb potential is given by (4.35), and the corresponding electric field \mathbf{E} by (4.34), with e replaced by e_b . The projectile therefore acquires a momentum transfer

$$\Delta \mathbf{p} = e_p \int_{-\infty}^{+\infty} dt \mathbf{E}(\mathbf{b} + \mathbf{v}_p t), \quad (4.37)$$

and it suffers a corresponding change in energy

$$\Delta E = \frac{\Delta \mathbf{p}^2}{2m_p}. \quad (4.38)$$

The component of the electric field along the direction of motion \mathbf{v}_p integrates to zero in (4.37), while the component normal to the trajectory gives the impulse

$$\Delta \mathbf{p} = e_p \int_{-\infty}^{+\infty} dt E_\perp(\mathbf{b} + \mathbf{v}_p t) \hat{\mathbf{b}} = \frac{e_p e_b}{\Omega_{\nu-1}} \int_{-\infty}^{+\infty} dt b (b^2 + v_p^2 t^2)^{-\nu/2} \hat{\mathbf{b}} \quad (4.39)$$

$$\sim \frac{1}{b^{\nu-2}} \hat{\mathbf{b}}. \quad (4.40)$$

¹⁴ In this simple example, I am not concerned with hyperbolic orbit corrections and the like; but rather, I am tracing the origin of the logarithmic divergence of the Coulomb potential and the pole at $\nu = 3$. For this, we can work with a hot dilute plasma where a linear trajectory will suffice.

The temperature equilibration rate contains a factor involving the cross-section weighted energy transfer, and we see that the energy exchange between p and a can be written

$$\frac{dE}{dt} \sim \int d\sigma \Delta E \sim \int_{b_{\min}}^{b_{\max}} \frac{db}{b^{\nu-2}} , \quad (4.41)$$

where we have used the fact that $d\sigma \cdot \Delta E \sim b^{\nu-2} db \cdot b^{-2(\nu-2)} \sim db/b^{\nu-2}$. I will elaborate further on this example in the next lecture, but for now one should simply note that the rate (4.41) implies that large ν is dominated by short-distance physics and small ν is dominated by long-distance physics. Moreover, expression (4.41) gives $\nu = 3$ as the dividing line between these two regions. To see this, note that for $\nu > 3$ the impact parameter integral is not sensitive to the large distance cut off, and we may simply take the limit $b_{\max} \rightarrow \infty$ to obtain

$$\nu > 3 : \quad \int_{b_{\min}}^{\infty} \frac{db}{b^{\nu-2}} = \frac{b_{\min}^{3-\nu}}{\nu-3} \Rightarrow \text{UV dominant and pole at } \nu = 3 . \quad (4.42)$$

Conversely, for $\nu < 3$, we can set $b_{\min} = 0$, with

$$\nu < 3 : \quad \int_0^{b_{\max}} \frac{db}{b^{\nu-2}} = \frac{b_{\max}^{3-\nu}}{3-\nu} \Rightarrow \text{IR dominant and pole at } \nu = 3 . \quad (4.43)$$

The results displayed are the dominant forms in the two regions of spatial dimensionality about $\nu = 3$. In either case, the stopping power contains a pole $1/(\nu - 3)$.

There are a number of important consequences arising from the UV and IR behaviors of the Coulomb potential (4.35). I will discuss this more fully in Section VI A, but for now recall that the derivation of the Boltzmann equation, as presented in Ref. [14] for example, breaks down for the Coulomb potential in three spatial dimensions because of the aforementioned infrared singularity. However, in $\nu > 3$ the “textbook derivation” of the Boltzmann equation with the Coulomb potential (4.35) is finite and completely rigorous. The simple pole at $\nu = 3$ in the scattering kernel corresponds to an IR divergence because of the long-range nature of the Coulomb force in three dimensions. Furthermore, because dimensions greater than three enhance the UV physics, the classical Born, Bogoliubov, Green, Kirkwood, and Yvon (BBGKY) hierarchy reduces to the Boltzmann equation to leading order in g (or to leading order in the number density) when $\nu > 3$. A similar reduction from BBGKY holds for the Lenard-Balescu equation in $\nu < 3$, and the “textbook derivation” [15] is also rigorous in these dimensions. In $\nu = 3$, the derivations of the Boltzmann and Lenard-Balescu equations break down for the Coulomb potential. This is not because the $\nu = 3$ version of the BBGKY hierarchy is divergent, but because the truncation procedure that leads to the Boltzmann and Lenard-Balescu equations breaks down for the three dimensional Coulomb potential.¹⁵ Indeed, BBGKY is completely finite for the Coulomb potential in $\nu = 3$, albeit completely useless for our purposes. Section II of BPS [1] provides more details, especially the two paragraphs between Eqs. (2.7) and (2.8).

¹⁵ For example, in deriving the Boltzmann equation from BBGKY, one invokes a principle of uncorrelated

V. THE LAMB SHIFT AND THE COULOMB LOGARITHM

The Lamb shift is interesting for us because it provides another connection with the Coulomb logarithm, both in the historical details and in much of the physics. The Lamb shift is a small energy split in the otherwise degenerate $2S$ and $2P$ states of total angular momentum $j = 1/2$. The measured value is about $\Delta E_{\text{lamb}} \simeq 4.4 \times 10^{-6} \text{ eV}$, with the $2S_{1/2}$ state lying above the $2P_{1/2}$ state. Therefore, when an electron makes a transition from the S -state to the P -state, it emits a microwave photon of frequency $\Delta \nu_{\text{lamb}} \simeq 1060 \text{ MHz}$. Calculating the observed value of the Lamb shift was the first great success of quantum electrodynamics (QED), the relativistic quantum theory of light and matter.¹⁶

Dirac's relativistic model of the hydrogen atom, *i.e.* his relativistic theory of the electron in a Coulomb potential, made the prediction that the energy levels of the hydrogen atom (neglecting the hyperfine structure) depend only upon the principal quantum number n and the total angular momentum j (the sum of the orbital momentum ℓ and the spin $s = 1/2$ of the electron). In particular, Dirac predicted that the $2S_{1/2}$ and $2P_{1/2}$ states should be degenerate. At the 1947 Shelter Island conference in New York, W. Lamb and R. Retherford announced the results of their highly sensitive experiment measuring the emission frequency of photons in a $2S$ - $2P$ transition, thereby establishing the unequivocal experimental existence of the Lamb shift.¹⁷ While almost degenerate compared to the binding energy of the hydrogen atom, today's accepted experimental splitting is [16]

$$\Delta E_{\text{lamb}} \equiv E_{2S_{1/2}} - E_{2P_{1/2}} = 4.374898(7) \times 10^{-6} \text{ eV} \quad (5.1)$$

$$\Delta \nu_{\text{lamb}} \equiv \frac{\Delta E_{\text{lamb}}}{h} = 1057.845(9) \text{ MHz} , \quad (5.2)$$

where Plank's constant in the form $h = 4.135\,667\,33(10) \times 10^{-15} \text{ eV-s}$ is the conversion factor between energy and frequency. Hans Bethe had attended the Shelter Island conference, and

collisions to replace the two-point correlation function by the product of two one-point functions, and this leads to the infrared divergence. This means, of course, that it is the long-distance correlations themselves that tame the IR divergence of the Boltzmann equation, correlations that are neglected by the truncation process. As a related point, this procedure imposes an implicit initial condition, thereby providing a *direction of time* for the Boltzmann equation (and this occurs even for short-range potentials).

¹⁶ The other early success of QED, which followed soon after the Lamb shift, was calculating the magnetic dipole moment of the electron. It is customary to write the magnetic moment of the electron μ_e in terms of the Bohr magneton $\mu_B = e\hbar/2m_e$ by introducing the dimensionless g -factor: $\mu_e = g_e \mu_B$. Using relativistic single-particle quantum mechanics, in 1928 Dirac predicted $g_e = 2$ (exactly). In 1948 Schwinger used QED to calculate the radiative corrections, and he found $g_e/2 = 1 + \alpha/2\pi + \mathcal{O}(\alpha^2) = 1.0011614$, which was in excellent agreement with experiment. Today, the electron's magnetic dipole moment has been calculated to include α^4 terms, and is in agreement with experiment to 10 significant figures, the most accurately verified quantity in the history of physics.

¹⁷ In the 1930s, S. Pasternak analyzed experimental data suggesting that such an energy split might exist; however, the systematic error was as large as the energy splitting itself.

on the train ride back he performed the first calculation of the Lamb shift, finding the value $\Delta\nu_{\text{bethe}} = 1040 \text{ MHz}$. Bethe's calculation was recognized as being only a rough approximation, neglecting high energy relativistic effects, but its close agreement with experiment was cause for optimism.

The dominant contribution to the Lamb shift comes from the bound $2S$ and $2P$ electrons exchanging virtual photons with the atomic nucleus.¹⁸ These radiative corrections effectively smear the point-like nature of the nucleus, thereby altering the atomic energy levels (the electron no longer “sees” a pure Coulomb potential). Any possible number of photon exchanges with the nucleus are permitted, from a single high-energy photon (hard/UV physics) to many low-energy photons (soft/IR physics), and this means there are two disparate but competing energy scales in the problem. In a manner similar to the Coulomb energy-loss rate (2.1), the Lamb shift takes the form

$$\Delta E_{\text{lamb}} = K \ln \left\{ \frac{E_{\text{max}}}{E_{\text{min}}} \right\} . \quad (5.3)$$

The UV scale is set by $E_{\text{max}} \sim m_e c^2 = 511 \text{ keV}$ and the IR scale E_{min} is set by the binding energy of the hydrogen atom

$$E_0 = \frac{1}{2} \left(\frac{e^2}{4\pi\hbar} \right)^2 m_e = \frac{1}{2} \alpha^2 m_e c^2 = 13.6 \text{ eV} . \quad (5.4)$$

To understand the origin of these scales, note that E_{max} takes its value from the energy at which relativistic effects for the electron become important (the rest-mass energy of the electron), while E_{min} takes its value from the only low-energy scale in the problem (namely, the binding energy of the atom). The soft photon IR interactions can be handled by nonrelativistic means, while the hard UV photons require a more complicated relativistic treatment. For this reason, Bethe used the simpler nonrelativistic formalism, cutting his calculation off at the relativistic energy scale $E_{\text{max}} \sim m_e c^2$ at which his formalism broke down. This is akin to cutting the calculation of the energy exchange rate off at some small distance scale b_{min} . Bethe then concentrated on the low energy theory, which is analogous to looking at the Lenard-Balescu equation in the rate calculation. Bethe found a logarithmic UV divergence, in the same way the Lenard-Balescu equation has a UV divergence, but he was able to regularize the infinity by applying a technique known as mass renormalization, thereby rendering his calculation finite. His calculation was completely rigorous at low energies, and Bethe found the coefficient K and the low energy cutoff E_{min} , both *exactly* [17]:

$$K = \frac{\alpha^3}{3\pi} E_0 \quad \text{and} \quad E_{\text{min}} = A_0 E_0 \quad \text{with} \quad E_{\text{max}} \sim m_e c^2 . \quad (5.5)$$

¹⁸ There are also several other subdominant mechanisms, such as the vacuum polarization of the photon and the anomalous magnetic dipole moment of the electron. These give, respectively, the contributions $\Delta\nu_{\text{lamb}}^{\text{vac}} = -27 \text{ MHz}$ and $\Delta\nu_{\text{lamb}}^{\text{mag}} = +68 \text{ MHz}$.

The binding energy E_0 of the hydrogen atom is given by (5.4), and while I will not write it down, the coefficient A_0 is rigorously defined in terms of sums of matrix elements of the various intermediate states. These matrix elements are sufficiently complicated that one can only calculate them numerically, and to three significant figures Bethe found $A_0 = 17.8$. Bethe's exact calculation of E_{\min} is akin to an exact calculation of b_{\max} in the rate problem. This, however, is a point where the analogy is not precise: Bethe was able to exactly calculate E_{\min} through the regularization procedure of mass renormalization, which has no counterpart in plasma physics. On the other hand, Lyman Spitzer was only able to estimate the value of the maximum impact parameter in the Coulomb logarithm to be of order $b_{\max} \sim \kappa_D^{-1}$. Bethe was able to perform this feat because QED is a fundamental theory of nature, while the Boltzmann equation is not. Using (5.4), we can express (5.3) as

$$\Delta E_{\text{lamb}}^{\text{B}} = -\frac{\alpha^5}{3\pi} m_e c^2 \ln\{C_{\text{B}} \cdot \alpha\} \quad \text{with} \quad C_{\text{B}} = \sqrt{17.8/2} = 2.98, \quad (5.6)$$

where $\alpha = e^2/4\pi\hbar c = 1/137.036$ is the fine structure constant. Bethe's calculation therefore gives $\Delta E_{\text{lamb}}^{\text{B}} = 4.3 \times 10^{-6} \text{ eV}$ or $\Delta\nu_{\text{lamb}}^{\text{B}} = 1040 \text{ MHz}$.

Bethe did not, however, calculate the *exact* coefficient under the logarithm. Since his calculation broke down at relativistic energies, he used the somewhat *ad hoc* value $E_{\max} = m_e c^2$ (rather than some multiple thereof) for the UV cutoff. It turns out that the high energy corrections to (5.6) are rather small, so Bethe's result was perhaps more accurate than warranted. Since Bethe only estimated the maximum energy cutoff, his result (5.6) is only accurate to leading order in α , and the constant under the logarithm required a more accurate treatment. The analogy further continues: Bethe calculated the Lamb shift to leading order, just as Spitzer calculated the Coulomb logarithm to leading order. Both calculations found the correct coefficient in front of the logarithm, and the order of magnitude of the argument of the logarithm. These calculations failed to extract the exact constant under the logarithm, although Bethe managed to find the exact expression for the low-energy cutoff. In this sense, Bethe is the Spitzer of the Lamb shift.

Shortly after this, R. Feynman and J. Schwinger independently calculated the high energy contribution using their respective relativistically covariant formalisms,¹⁹ and their calculations were exact to leading and next-to-leading order in α . After adding their contribution of the high energy corrections to Bethe's low energy form, the calculation of the Lamb shift was complete and the constant under the logarithm was fully determined. Or so it would

¹⁹ F. Dyson soon proved that Schwinger's mathematically rigorous formalism was equivalent to Feynman's more intuitive but easier to use approach. Along with the Japanese physicist S. Tomonaga, Schwinger and Feynman shared the 1965 Nobel prize "for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles."

seem. Simultaneously, J. French and V. Weisskopf completed an independent calculation of the Lamb shift that disagreed with Feynman-Schwinger, albeit only slightly in the coefficient under the logarithm. The plot thickened as yet another calculation of the Lamb shift was completed that agreed with French-Weisskopf, this time from N. Kroll and Lamb himself.²⁰ So now we seem to have a real problem: the Schwinger-Feynman calculations agree with each other, but disagree with the French-Weisskopf and Kroll-Lamb calculations. Schwinger and Feynman had developed independent but equivalent formalisms that manifestly exhibited the relativistic covariance of the theory, while the other four had used a cumbersome formalism developed in the 1930's (now called *old-fashioned perturbation theory*). As it turns out, Schwinger and Feynman had made the same subtle mistake, and Weisskopf's calculation was correct. The contribution to the Lamb shift we have been considering has the form

$$\Delta E_{\text{lamb}} = \frac{\alpha^3}{3\pi} E_0 \left(\ln \left\{ \frac{m_e c^2}{2E_B} \right\} + \frac{91}{120} \right) , \quad (5.7)$$

where $E_B = 17.8E_0$ is Bethe's low energy result (5.5). As with the BPS Coulomb logarithm, we can bring the additive constant in (5.7) inside the logarithm, and using (5.4) we can express (5.7) as

$$\Delta E_{\text{lamb}} = -\frac{\alpha^5}{3\pi} m_e c^2 \ln\{C \alpha\} \quad \text{with } C = 2.98 . \quad (5.8)$$

In terms of the Bethe's coefficient C_B of (5.6), the complete Lamb shift coefficient is $C = \sqrt{2} e^{-91/240} C_B$ (apart from the photon vacuum polarization and the electron dipole moment corrections, which were mentioned in footnote 18, but will not concern us further).

French spent the next year tracing down the origins of the discrepancy between the calculations. He could not find an error in the Feynman-Schwinger high energy calculation, nor was there an error in Bethe's low energy calculation. Instead, the error was in the way the high-energy calculation of Feynman-Schwinger was "joined" onto the low-energy result of Bethe. The high-energy calculation is also infinite and in need of regularization, but this time it contains an IR divergence. This is analogous to the Boltzmann equation containing the correct short-distance physics, but nonetheless diverging at long distances. Therefore, Feynman-Schwinger had to introduce an intermediate step in which they regulated their high energy theories in the infrared. They chose to do this by assigning a small mass to the photon, and then taking this mass to zero at the end of the calculation. While this is a common IR regularization scheme in QED, it is incommensurate with Bethe's regularization scheme in his low energy calculation, where he used a simple cutoff procedure in a high-energy integral.

²⁰ Lamb was both an experimentalist and a theorist, and he received the 1955 Nobel Prize for his "discoveries concerning the fine structure of the hydrogen spectrum."

French-Weisskopf and Kroll-Lamb had gotten the correct result because they calculated both the low-energy and the high-energy contributions using the *same* formalism, and therefore with consistent regularization schemes for both short and long distances. For more on the consequences of regulating large and small scales in an incommensurate manner, see Feynman's footnote 13 in Ref. [18] of this lecture, which I have quoted in the bibliography in its entirety. That one must calculate the large and small scales in exactly the same manner is not a minor point, as underscored by the stature of the physicists who failed to realize its importance when they first calculated the Lamb shift from the then fledgling theory of quantum electrodynamics (QED).

There are a number of consistent regularization schemes in use today in QED and other field theories of nature, with dimensional regularization being one of the most popular and easy to use. These regularization and renormalization schemes have allowed us to calculate a great many experimentally verified quantities, to extremely high precision, and there is no longer any doubt in their correctness. While extending the dimension of space to complex values might at first seem unsettling, this procedure works. When dimensional continuation was first introduced into quantum field theory, there were strong reactions against it. However, calculations performed with this method agree with calculations using other regularization schemes, and more important, with experiment. In time, particle physicists learned to accept the notion that one can perform correct three dimensional calculations by working in arbitrary complex dimensions. For our concerns, we note that Refs. [8] and [16] have indeed calculated the Lamb shift using the method of dimensional continuation. The problem of Feynman-Schwinger is avoided, and dimensional continuation gives the correct experimentally observed result with much less effort than more traditional methods. As the above discussion illustrates, this is no small achievement.

In summary, dimensional continuation is powerful because (i) it is a consistent regularization scheme that (ii) lends itself to a perturbative analysis, and (iii) requires relatively simple (or at least easily learned) calculational tools. DeWitt's calculation [7] was certainly consistent, in that it matched the long and short-distance physics commensurately [to order $\mathcal{O}(g^3)$]. This is because he starts with a finite regulated theory that treats long- and short-physics together within a single framework, albeit with a regularization scheme that does not lend itself to a systematic perturbative analysis. By comparison with the BPS result, we know DeWitt was accurate to order g^2 , inclusive. But, as illustrated by Feynman's footnote 13 of Ref. [18], and by the story of the Lamb shift above, any attempt at treating the long- and short-distance physics by separate regularization schemes (as much of the plasma literature currently does), will likely miss the very constants they are trying to calculate. Recall, it took two correct calculations to find the error in Feynman's single calculation. Dimensional continuation performs all necessary book keeping, at both small and large scales, and in a manner that affords simple calculations and perturbative expansions.

VI. CALCULATING THE RATE SYSTEMATICALLY WITH DIMENSIONAL CONTINUATION

A. Dimensional Reduction of BBGKY

Let us return to the rate equation (1.7) in the light of the apparatus of dimensional continuation that we have constructed. Since we are interested in spatially uniform plasmas, we will only consider particle distributions that are functions of the momentum, or equivalently the velocity. Let \mathbf{v}_ν denote a ν -dimensional velocity vector with components v_ℓ for $\ell = 1, \dots, \nu$, and define a ν -dimensional distribution function $f_\nu(\mathbf{v}_\nu, t)$ by

$$d^\nu v f_\nu(\mathbf{v}_\nu, t) \equiv \text{number of particles in a hyper-volume } d^\nu v \text{ about } \mathbf{v}_\nu \text{ at time } t. \quad (6.1)$$

Then the generalization of the three dimensional result (1.7) to ν -dimensions would be

$$\frac{d\mathcal{E}_\nu}{dt} = \int d^\nu v \frac{1}{2} m v_\nu^2 \frac{\partial f_\nu}{\partial t}(\mathbf{v}_\nu, t), \quad (6.2)$$

where the square of the velocity in (6.2) is $v_\nu^2 = \mathbf{v}_\nu \cdot \mathbf{v}_\nu = \sum_{\ell=1}^\nu v_\ell^2$. As previously mentioned, the standard textbook calculation of the Boltzmann equation goes through without an infrared divergent scattering kernel when $\nu > 3$. For now, I will write this equation in schematic form as

$$\frac{\partial f_\nu}{\partial t} + \mathbf{v}_\nu \cdot \nabla f_\nu = B_\nu[f] \quad : \nu > 3. \quad (6.3)$$

Note that ∇ is the ν -dimensional gradient and $\mathbf{v}_\nu \cdot \nabla f_\nu = \sum_{\ell=1}^\nu v_\ell \partial f_\nu / \partial x_\ell$. In dimensions $\nu < 3$, the Lenard-Balescu equation is ultraviolet finite, and we have

$$\frac{\partial f_\nu}{\partial t} + \mathbf{v}_\nu \cdot \nabla f_\nu = L_\nu[f] \quad : \nu < 3. \quad (6.4)$$

The explicit form of the scattering kernels $B_\nu[f]$ and $L_\nu[f]$ will not be required until the next lecture. As we shall see, the scattering kernels B_ν and L_ν are simply the obvious generalizations of their three dimensional counter parts: momentum and wave-number vectors live in ν -dimensions, and the scattering is produced by the ν -dimensional Coulomb potential (4.35). Calculations using (6.3) in $\nu > 3$ and (6.4) in $\nu < 3$, respectively, are completely finite.

In exactly the same manner, one can generalize the BBGKY hierarchy to an arbitrary number of dimensions, and this will be the starting point for my treatment of dimensional continuation. As we discussed in the closing paragraph of the last section, when the number of spatial dimensions is greater than three, BBGKY reduces to the Boltzmann equation (6.3)

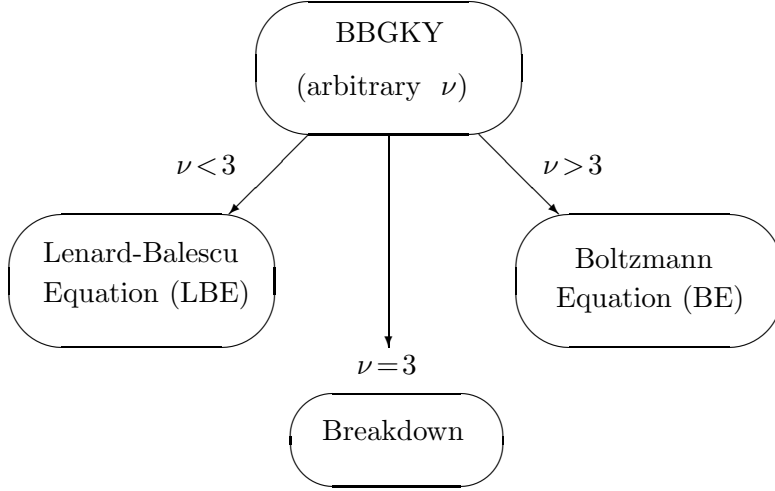


FIG. 4: For $\nu > 3$ the “textbook derivation” of the Boltzmann equation for a Coulomb potential is rigorous; furthermore, the BBGKY hierarchy reduces to the Boltzmann equation to leading order in g . A similar reduction from the BBGKY hierarchy holds for the Lenard-Balescu equation in $\nu < 3$, and the “textbook derivation” is also rigorous in these dimensions. In $\nu = 3$, the *derivations* of the Boltzmann and Lenard-Balescu equations break down for the Coulomb potential. This is not because the three dimensional BBGKY hierarchy breaks down, but because the Boltzmann and Lenard-Balescu equations break down. Indeed, BBGKY is completely finite for the Coulomb potential in $\nu = 3$, albeit completely useless for our purposes.

to leading order in the plasma coupling g . Therefore, when $\nu > 3$, to leading order in g the rate becomes

$$\frac{d\mathcal{E}_\nu^>}{dt} = \int d^\nu v \frac{1}{2} m v_\nu^2 B_\nu[f] \quad : \quad \nu > 3 . \quad (6.5)$$

Conversely, in dimensions $\nu < 3$, BBGKY reduces to the Lenard-Balescu equation (6.4) to leading order in g , and

$$\frac{d\mathcal{E}_\nu^<}{dt} = \int d^\nu v \frac{1}{2} m v_\nu^2 L_\nu[f] \quad : \quad \nu < 3 . \quad (6.6)$$

This is illustrated in Fig. 4. The “greater-than” and “less-than” superscripts are to remind us that the integrals in the rate calculations (6.5) and (6.6) are to be preformed in $\nu > 3$ and $\nu < 3$, respectively. Because of spatial uniformity, I have set the convective terms in (6.3) and (6.4) to zero, $\mathbf{v}_\nu \cdot \nabla f_\nu = 0$. Using the appropriate Coulomb potential (4.35) for $V_\nu(r)$ in the scattering kernels of (6.5) and (6.6), the integrals now converge, and they are calculated exactly in Sections 7 and 8 of BPS [1]. In the next lecture, we will calculate these integrals in the Born approximation.

Note that this is a first-principles derivation of the rates (6.5) and (6.6) in their respective dimensions ν . Let me reiterate the argument once again, although with a slightly different

emphasis. For simplicity, we consider the purely classical regime first, adding quantum mechanics in a moment. The classical BBGKY hierarchy for the Coulomb potential in three spatial dimensions is well defined and finite. We run into trouble only when we attempt to truncate the hierarchy and derive lower-order kinetic equations, such as the Boltzmann and the Lenard-Balescu equations. The necessity of truncated equations is of course clear (even the three-body problem cannot be solved analytically). Unfortunately, however, the Coulomb potential in three dimensions produces divergent scattering kernels in these truncated equations. Rather than creating a model of the ostensibly divergent scattering kernel, we shall instead systematically regulate the divergences by letting the spatial dimension depart from its empirically measured value of three. Logarithmic divergences in three dimensions then become simple poles of the form $1/(\nu - 3)$ in arbitrary dimensions. As with the regularization procedure of quantum field theory, our starting point here is a *well-defined and finite theory*, albeit in ν dimensions, *regularized in a consistent fashion at all length and energy scales*.

Let us now return to classical BBGKY, but this time in a spatial dimension ν of arbitrary positive *integer* value (we are not yet considering continuous values of ν). At first sight, the hierarchy equations in ν dimensions are just as useless as those in three dimensions — there are simply too many of them to solve. However, if we are willing to work to leading order accuracy in the plasma coupling g , which is quite accurate for a weakly to moderately coupled plasma, then: (i) in $\nu > 3$ we can truncate BBGKY to the ν -dimensional Boltzmann equation, and (ii) in $\nu < 3$ we can truncate BBGKY to the ν -dimensional Lenard-Balescu equation. Quantum scattering effects in the plasma will not modify the ν -dimensional Lenard-Balescu equation, but they will modify the ν -dimensional Boltzmann equation. Since we require the leading order term in the rate to be exact, we must include two-body quantum scattering effects exactly (but no more than two-body effects, since these are subleading in g — in fact, three-body and higher correlations and scattering effects, both classical and quantum, can and *should* be neglected to leading order in g). Two-body quantum effects can be accounted for by replacing the classical cross section in the Boltzmann scattering kernel with the two-body quantum cross section (this can be performed *exactly* since all scattering phase shifts δ_ℓ are known for the Coulomb potential in three dimensions). These ideas will be illustrated in complete detail in the next lecture for a particularly useful but simple case.

B. Calculating the Leading Order Term

We now return specifically to the electron-ion temperature equilibration rate of (1.2). To obtain the leading order in g behavior when $\nu > 3$, we calculate the rate using (6.5). As our calculation in the next lecture will reveal, this rate is proportional to g^2 (or the number density) and takes the form²¹

$$\frac{d\mathcal{E}_{ei}^>}{dt} = H(\nu; \eta) \frac{g^2}{\nu - 3} + \mathcal{O}(\nu - 3) \quad : \text{ LO in } g \text{ when } \nu > 3, \quad (6.7)$$

where, for definiteness, we have restored the electron-ion subscript to the rate, as in (1.1). In the next lecture we shall calculate $H(\nu; \eta)$ explicitly, but for now it will suffice to note that H depends upon the spatial dimension ν and the quantum parameter η . The simple pole at $\nu = 3$ reflects the long-distance or infrared divergence of the Coulomb potential in three spatial dimensions, and it arises from an integral over the radial coordinate of the ν -dimensional Coulomb potential (4.35). In a similar manner, the leading order behavior in dimensions $\nu < 3$ is given by (6.6). In the next lecture, we will see that this takes the form

$$\frac{d\mathcal{E}_{ei}^<}{dt} = G(\nu) \frac{g^{\nu-1}}{3 - \nu} + \mathcal{O}(3 - \nu) \quad : \text{ LO in } g \text{ when } \nu < 3. \quad (6.8)$$

There is no η -dependence in $G(\nu)$ since the leading order long-distance physics is purely classical. In the next lecture, we will perform the integrals to establish (6.7) and (6.8), thereby calculating the coefficients $H(\nu)$ and $G(\nu)$ exactly (for notational simplicity, I will hereafter drop the η -dependence from H). Note that (6.8) also contains a simple pole at $\nu = 3$ arising from an integration of (4.35), but this time the pole corresponds to missing short-distance physics or the ultraviolet divergence of the Lenard-Balescu equation in three dimensions. In general, if a three-dimensional integral diverges, then the corresponding ν -dimensional integral will typically contain a simple pole of the form $1/(\nu - 3)$. In this way we can transform a divergent integral into a convergent quantity that we can manipulate, and this is how we shall regularize the divergent Boltzmann and Lenard-Balescu equations in three dimensions.

²¹ As we have already pointed out in footnote 9, the parameter $g = e^2 \kappa_D / 4\pi T$ given by (4.2) is dimensionless only for $\nu = 3$. From definition (4.1) and the Coulomb potential (4.35), the proper *dimensionless* expansion parameter should be $g_\nu = e^2 \kappa_D^{\nu-2} / C_\nu T$. However, since $g_\nu = g \cdot \kappa_D^{\nu-3} (4\pi / C_\nu)$, we can absorb factors of $\kappa_D^{\nu-3} (4\pi / C_\nu)$ into any accompanying coefficient, such as $H(\nu; \eta)$ in (6.7). Therefore, powers of the dimensionless coupling g_ν also count powers of the three dimensional coupling g . We are therefore free to think of the expansion, even in ν dimensions, in terms of the three dimensional parameter g .

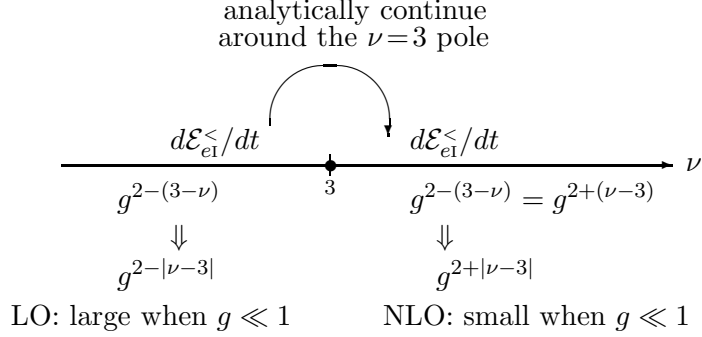


FIG. 5: The analytic continuation of $d\mathcal{E}_{el}^</dt$ from $\nu < 3$ to the region $\nu > 3$: the same expression can be used for $d\mathcal{E}_{el}^</dt$ throughout the complex plane since the pole at $\nu = 3$ can easily be avoided. Note that the quantity $d\mathcal{E}_{el}^</dt \sim g^{2+(\nu-3)}$ is leading order in g for $\nu < 3$. However, upon analytically continuing to $\nu > 3$ we find that $d\mathcal{E}_{el}^</dt \sim g^{2+|\nu-3|}$, which is next-to-leading order in g relative to $d\mathcal{E}_{el}^>/dt \sim g^2$.

C. Next-to-Leading Order from Leading Order via Analytic Continuation

Since the rates $d\mathcal{E}_{el}^>/dt$ of (6.7) and $d\mathcal{E}_{el}^</dt$ of (6.8) were calculated in mutually exclusive dimensional regimes, one might think that they cannot be compared. However, even though (6.8) was originally calculated in $\nu < 3$ for integer values of ν , we can analytically continue the result to complex values of ν (in the same way that the factorial function, which operates on positive integers, can be generalized to the Gamma function, which operates on the whole complex plane). In fact, if we continue $d\mathcal{E}_{el}^</dt$ to real values of ν with $\nu > 3$, then we can directly compare (6.7) and (6.8). Upon writing the g -dependence of (6.8) as $g^{2+(\nu-3)}$, when $\nu > 3$ we see that (6.8) is higher order in g than (6.7). By power counting arguments, there are no powers of g in (6.7) between g^2 and $g^{\nu-1}$ for $\nu > 3$, and therefore (6.8) indeed provides the correct next-to-leading order term in g when the dimension is analytically continued to $\nu > 3$,

$$\frac{d\mathcal{E}_{el}^<}{dt} = -G(\nu) \frac{g^{2+(\nu-3)}}{\nu-3} + \mathcal{O}(\nu-3) \quad : \text{ NLO in } g \text{ when } \nu > 3, \quad (6.9)$$

where I have written the exponent of g as $2 + (\nu-3)$ rather than $\nu-1$. This is illustrated in Fig. 5.

As we shall see in the next section, the individual pole-terms in (6.7) and (6.9) will cancel giving a finite result when the leading and next-to-leading order terms are added. The resulting finite quantity will therefore be accurate to leading and next-to-leading order in g as the $\nu \rightarrow 3$ limit is taken. Alternately, we could have analytically continued (6.7) to dimensions $\nu < 3$, where it would become next-to-leading order relative to (6.8). In either case, the leading and next-to-leading order contribution can be found by simply adding (6.7) and (6.8) and then taking $\nu \rightarrow 3$.

D. Returning to Three Dimensions

To find the three dimensional rate $d\mathcal{E}_{\text{el}}/dt$, accurate to leading and next-to-leading order in the plasma coupling g , we add the leading order expression (6.7) for $d\mathcal{E}_{\text{el}}^>/dt$ to the next-to-leading order expression (6.9) for $d\mathcal{E}_{\text{el}}^</dt$, and then take the limit $\nu \rightarrow 3^+$:

$$\frac{d\mathcal{E}_{\text{el}}}{dt} = \lim_{\nu \rightarrow 3^+} \left[\frac{d\mathcal{E}_{\text{el}}^>}{dt} + \frac{d\mathcal{E}_{\text{el}}^<}{dt} \right] + \mathcal{O}(g^3) . \quad (6.10)$$

For the same reasons as given in Section III E, this does not lead to any form of “double counting.” Instead, we are simply adding the next-to-leading order term (6.9) to the leading order term (6.7) at a common value of $\nu > 3$.

Since (6.10) lies at the heart of dimensional continuation, allow me review the reasoning behind this expression one final time. Recall from (6.7) and (6.8) that $d\mathcal{E}_{\text{el}}^>/dt$ and $d\mathcal{E}_{\text{el}}^</dt$ are both leading order in g for $\nu > 3$ and $\nu < 3$, respectively. Since these functions were calculated for mutually exclusive values of ν , they must be analytically continued to the same value of ν for purposes of comparison. In equation (6.10), we have chosen to continue $d\mathcal{E}_{\text{el}}^</dt$ to $\nu > 3$, which takes the same algebraic form as it did for $\nu < 3$. The analytic continuation has the effect of rendering $d\mathcal{E}_{\text{el}}^</dt$ subleading in g relative to $d\mathcal{E}_{\text{el}}^>/dt$. There are no powers of g between these two terms for any value of ν , and therefore the limiting procedure $\nu \rightarrow 3^+$ gives the three dimensional rate *exactly* to leading and next-to-leading order accuracy in g . We have therefore found the leading order and next-to-leading order in g contributions for $\nu > 3$,

$$\frac{d\mathcal{E}_{\text{el}}^>}{dt} + \frac{d\mathcal{E}_{\text{el}}^<}{dt} = \frac{1}{\nu - 3} \left[H(\nu) g^2 - G(\nu) g^{2+(\nu-3)} \right] + \mathcal{O}(\nu - 3) . \quad (6.11)$$

I have not indicated the higher order error in g , but instead I am emphasizing here the error associated with the ν -expansion. That is to say, (6.11) contains an implicit error that approaches $\mathcal{O}(g^3)$ as $\nu \rightarrow 3^+$, while I have chosen to explicitly display the first-order error in $\nu - 3$. In the sum (6.11), the error in g remains nonzero in the three dimensional limit, while the ν -error in (6.11) vanishes as $\nu \rightarrow 3$.

In taking the $\nu \rightarrow 3$ limit, we must keep in mind that terms proportional to $\nu - 3$ will give a non-vanishing result when multiplied by the pole $1/(\nu - 3)$. Upon expanding the g -dependence to linear order in $\epsilon = \nu - 3$ we find

$$g^\epsilon = \exp\{\ln(g^\epsilon)\} = \exp\{\epsilon \ln g\} = 1 + \epsilon \ln g + \mathcal{O}(\epsilon)^2 , \quad (6.12)$$

which can be written as

$$\frac{g^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \ln g + \mathcal{O}(\epsilon) . \quad (6.13)$$

We must now expand the coefficients $H(\nu)$ and $G(\nu)$ in powers of $\epsilon = \nu - 3$. As we shall see in the next lecture, and this is a *crucial* point, the leading order in ϵ terms are identical, and so the expansions take the form

$$H(\nu) = -A + \epsilon H_1 + \mathcal{O}(\epsilon^2) \quad (6.14)$$

$$G(\nu) = -A + \epsilon G_1 + \mathcal{O}(\epsilon^2) . \quad (6.15)$$

In the next lecture, we will perform the integrals in (6.5) and (6.6), thereby allowing us to exactly compute $A \equiv H(\nu=3) = G(\nu=3)$ and the linear terms $H_1 \equiv H'(\nu=3)$ and $G_1 \equiv G'(\nu=3)$. The remaining procedure is now straightforward. Upon using (6.13)–(6.15) in (6.7) and (6.9), the leading and next-to-leading order terms in g now take the form

$$\frac{d\mathcal{E}_{\text{el}}^>}{dt} = -\frac{A}{\nu-3} g^2 + H_1 g^2 + \mathcal{O}(\nu-3; g^3) \quad (6.16)$$

$$\frac{d\mathcal{E}_{\text{el}}^<}{dt} = \frac{A}{\nu-3} g^2 - A g^2 \ln g - G_1 g^2 + \mathcal{O}(\nu-3; g^3) . \quad (6.17)$$

I have now indicated here the error associated with the g -expansion. Because the leading order terms in (6.14) and (6.15) are equal, the simple poles in (6.16) and (6.17) cancel. Therefore, the limit $\nu \rightarrow 3^+$ of expression (6.11) gives the rate

$$\frac{d\mathcal{E}_{\text{el}}}{dt} = -A g^2 \ln g + B g^2 + \mathcal{O}(g^3) , \quad (6.18)$$

with $B = H_1 - G_2$, in agreement with (4.7). In this way, BPS has calculated the energy exchange accurate to leading order and next-to-leading order in g .

VII. SOME CLOSING REMARKS

A. Summary

I hope this account of the simple field theoretic apparatus necessary for a reading of BPS has been helpful. Having understood the reasoning behind BPS, especially the manner by which analytic continuation turns a leading order result into a next-to-leading order result, we can summarize the BPS procedure by the following prescription:

1. Calculate the rate $d\mathcal{E}^>/dt$ in the regime $\nu > 3$ using the Boltzmann equation, including quantum corrections to all orders in η . This captures the leading order physics in dimensions greater than three.
2. Calculate the corresponding rate $d\mathcal{E}^</dt$ in the regime $\nu < 3$ using the Lenard-Balescu equation. This physics is classical, and captures the leading order behavior in dimensions less than three.

3. Add the terms $d\mathcal{E}^>/dt$ and $d\mathcal{E}^</dt$ and take the $\nu \rightarrow 3$ limit. This gives the energy exchange rate in three dimensions accurate to leading and next-to-leading order in g , $d\mathcal{E}/dt = -Ag^2 \ln\{C(\eta)g\} + \mathcal{O}(g^3)$. That is to say, this gives the coefficients A and C exactly.

This prescription provides complete analytic expressions for the coefficients A and $C(\eta)$, with $C(\eta)$ being of particular interest. In the language of the Coulomb logarithm, we write $\ln \Lambda_{\text{coul}}(g, \eta) = -\ln\{C(\eta)g\}$.

B. Further Context

The fundamental interactions of nature can be expressed as quantum field theories (except for gravity²², which to date has only been expressed as a classical field theory, *i.e.* general relativity). For this reason, quantum field theory reveals something very deep about the structure of nature. However, for our purposes, we only need to think of quantum field theory as an elegant collection of tools for packaging and solving many-body problems, and plasma physics concerns itself with the many-body problem par excellence. It should therefore come as little surprise that quantum field theory can be useful for plasma physics. Reference [1] is a nice example of cross fertilization between two quite different branches of physics, and it is gratifying for someone who has worked in both particle and plasma physics that such seemingly different subjects can inform one another. Perhaps the most dramatic difference between particle and plasma physics lies with their respective methods and outlooks, and not so much the subject matter itself. A number of plasma physicist have explained to me that that plasma physics is not an *exact* science. While the nature of the subject might render a first-principles approach limited, particularly for subtopics like magneto-hydrodynamics in tokamaks or self-organized behavior in strongly coupled plasmas, I believe the plasma physicist could still benefit from the more rigorous outlook of particle physics. Conversely, the particle physicist could benefit from plasma physics. In fact, until new high energy experiments come on-line, plasma physics may have more to offer particle physics than the reverse.²³ Until recently, high energy physics was a spectacular field in

²² There have of course been numerous attempts at quantizing gravity, but since none of these has yet produced a full fledged theory of quantum gravity, I do not count them here. In fact, it might not be possible to describe gravity with a field theory on the quantum level. This would be true, for example, if string theory were to provide the theory of quantum gravity, since a string theory is qualitatively different from a field theory.

²³ Particle physics is currently in crisis brought on from a dearth, and indeed a complete absence, of new experimental results. The only hope for high energy physics seems to lie with the Large Hadron Collider at CERN, or its possible successors. Compare this situation with the current Renaissance in observational cosmology and astrophysics, which has more and better quality data than at any time during its history.

which to work, but, with a few exceptions such as lattice gauge theory and RHIC physics at BNL, the field seems to have been milked dry. High energy physics is in dire need of direct experimental input. Plasma physics, on the other hand, still has many interesting and quite challenging problems of experimental and astrophysical relevance, and a number of these seem tailor-made for the field theorist. The particle physicist could therefore benefit by an excursion (or even a longer stay) in plasma physics. Hence, my hidden agenda in these lectures: it would be quite nice to entice a few particle theorists to work on some of the interesting problems of plasma physics. It would also be nice to convince a few plasma physicists that particle theory has something to offer them as well.

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Sadly, the field of high energy physics has responded to its data crisis in a rather pathological, although perhaps predictable, manner: the mono-culture of string theory and string theory "inspired" theories, none of which are empirically based, and all of which virtually dominate the entire landscape of high energy physics. At the 10% level, string theory and the like might be healthy (after all, a quantum description of gravity simply falls out of string theory, and it is hard to believe that this is an accident); however, at the current level, string theory and its kinfolk are as stifling as a hillside of kudzu in Tennessee.

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- [18] R. P. Feynman, *Space-Time Approach to Quantum Electrodynamics*, Phys. Rev. **76** (1949) 769, footnote 13, which I will quote in it entirety:

That the result given in B [R.P. Feynman, Phys. Rev. 74, 1430 (1948)] in Eq. (19) was in error was repeatedly pointed out to the author, in private communication, by V.F. Weisskopf and J.B. French, as their calculation completed simultaneously with the author's early in 1948 gave a different result. French has finally shown that although the expression for the radiationless scattering in B, Eq. (18) or (24) above is correct, it was incorrectly joined onto Bethe's non-relativistic result. He shows that the relation $\ln 2k_{\max} - 1 = \ln \lambda_{\min}$ used by the author should have been $\ln 2k_{\max} - 5/6 = \ln \lambda_{\min}$. This results in adding a term $-(1/6)$ to the logarithm in B, Eq. (19) so that the result now agrees with that of J.B. French and V.W. Weisskopf, Phys. Rev. 75, 1240 (1949) and N.H. Kroll and W.E. Lamb, Phys. Rev. 75, 388 (1949). The author feels unhappily responsible for the very considerable delay in the publication of French's result occasioned by this error. This footnote is appropriately numbered.